Approximate pure Nash equilibria

in weighted congestion games

Max Klimm

Combinatorial Optimization and Graph Algorithms
Technische Universität Berlin

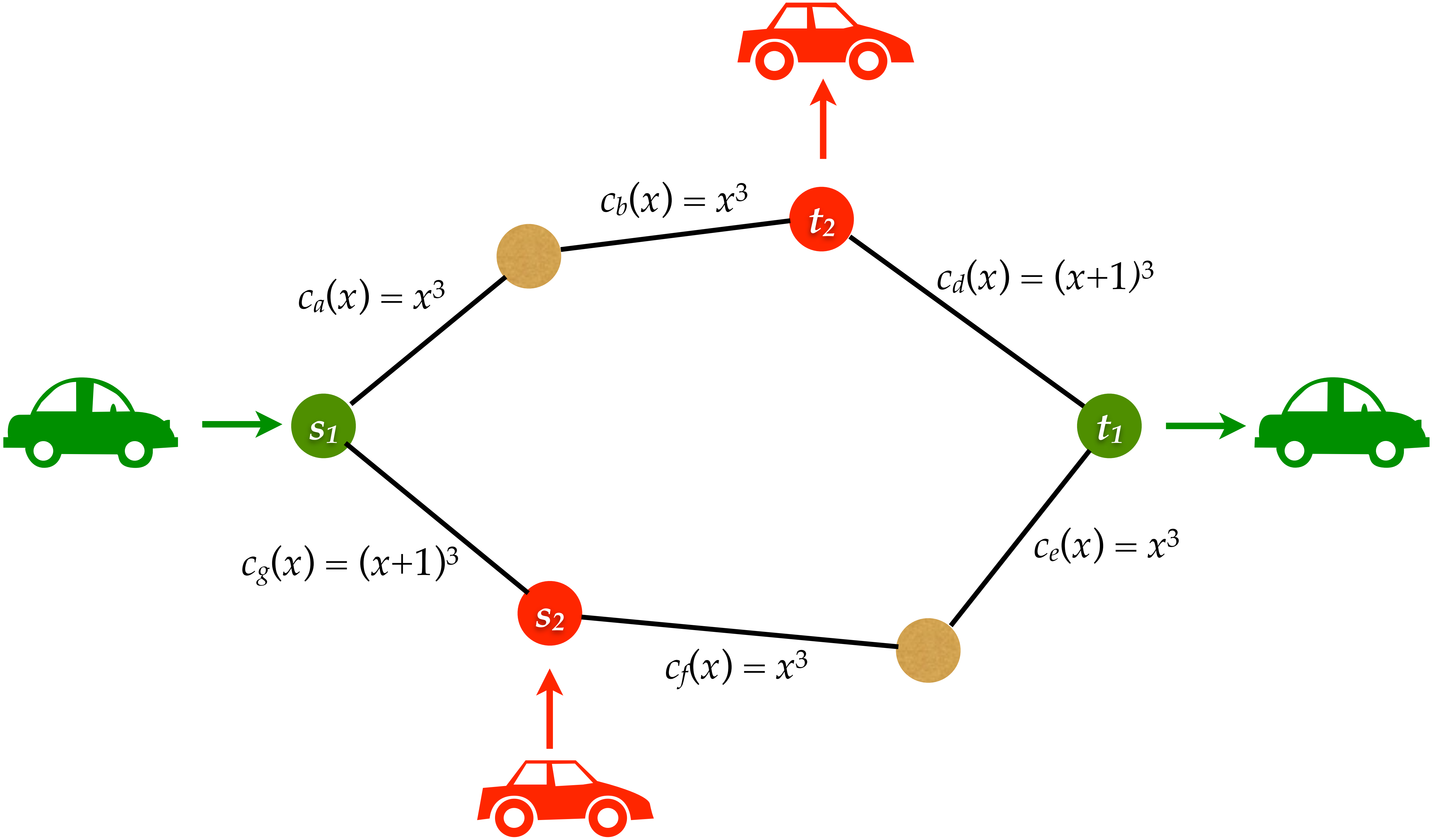
Christoph Hansknecht

Combinatorial Optimization and Graph Algorithms
Technische Universität Berlin

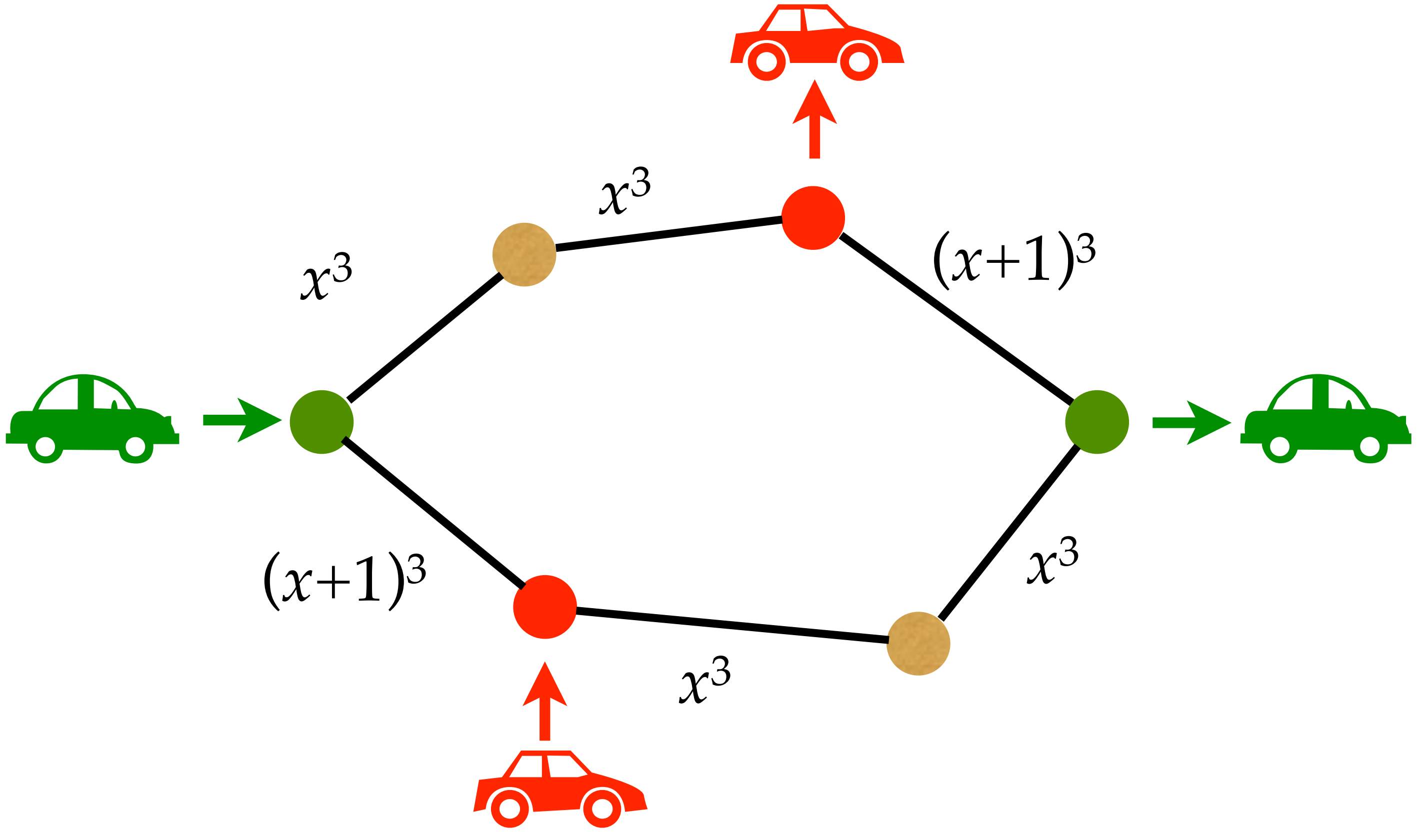
Alexander Skopalik

Heinz Nixdorf Institute
University of Paderborn

Introduction

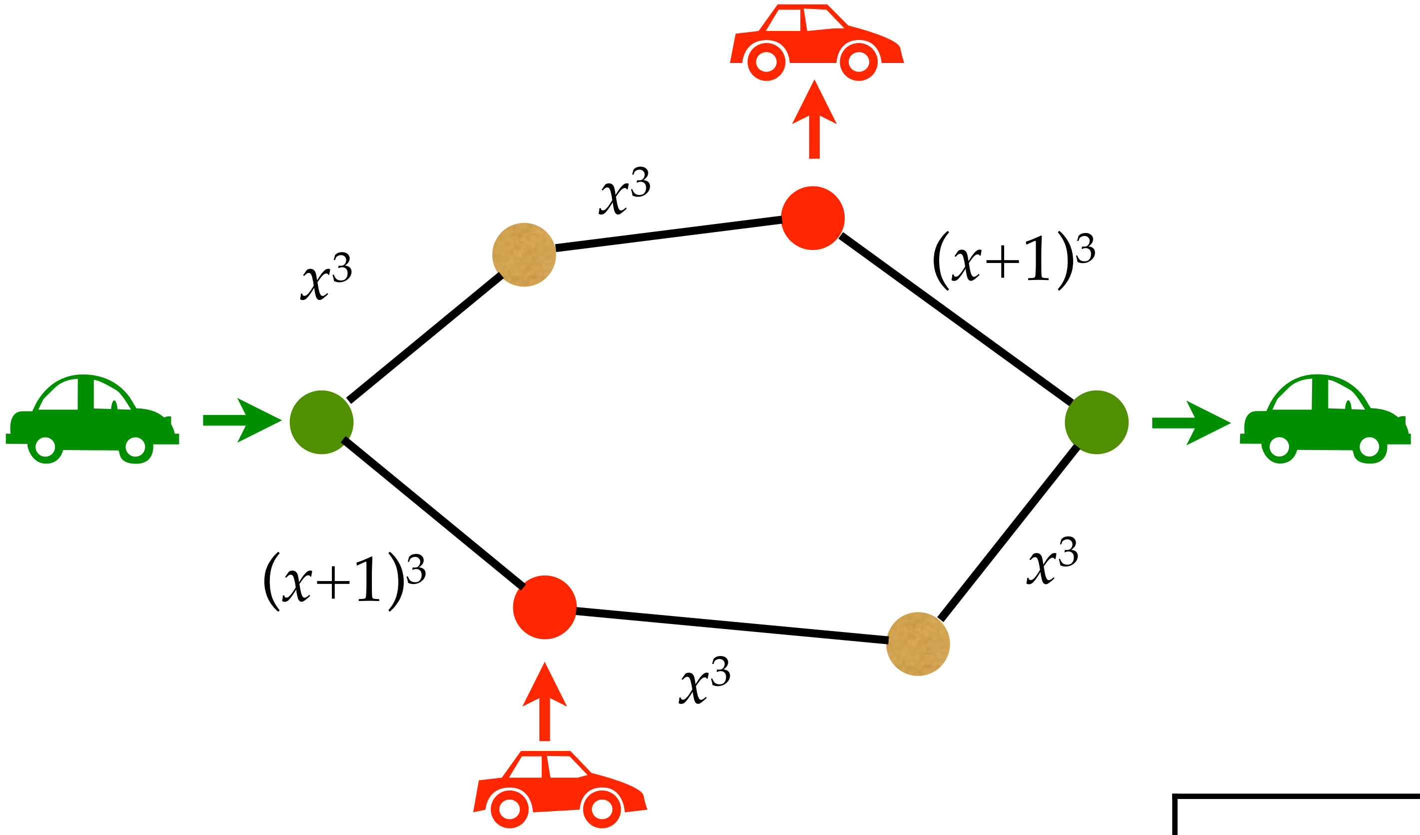


Introduction

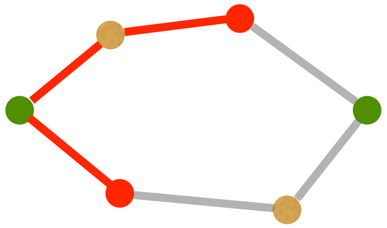
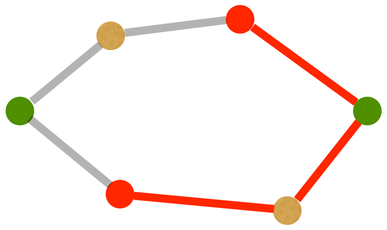
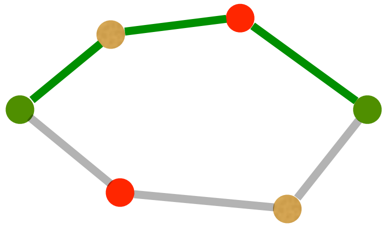
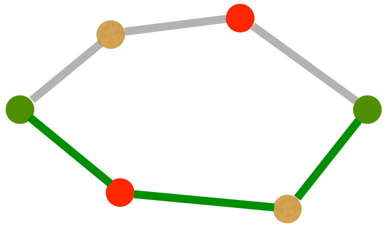


Nash equilibrium
Strategy combination such that no player can unilaterally improve

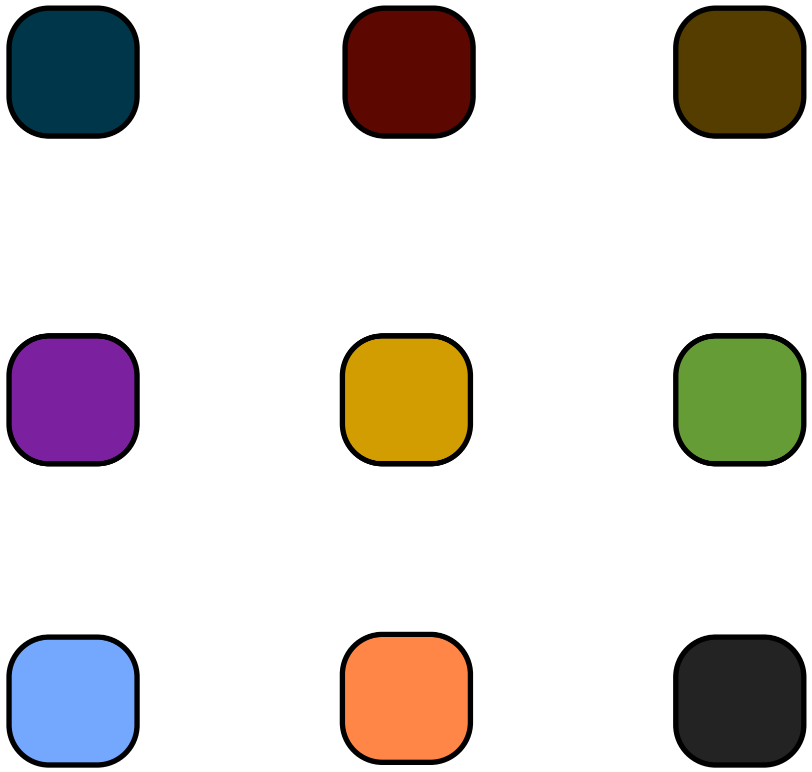
Introduction



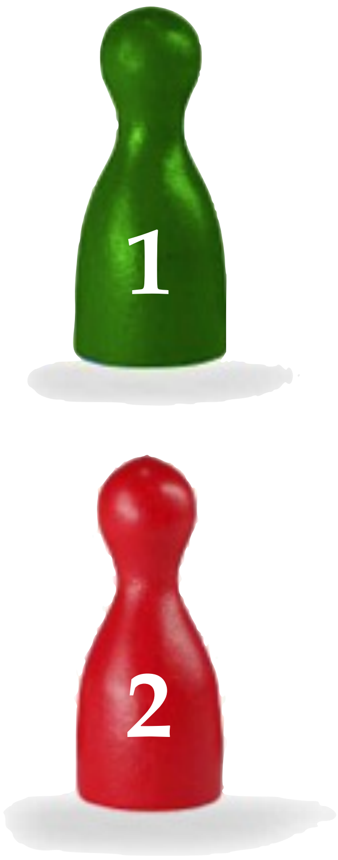
Nash equilibrium
Strategy combination such that no player can unilaterally improve

		
	24, 24	29, 29
	29, 29	24, 24

Congestion games

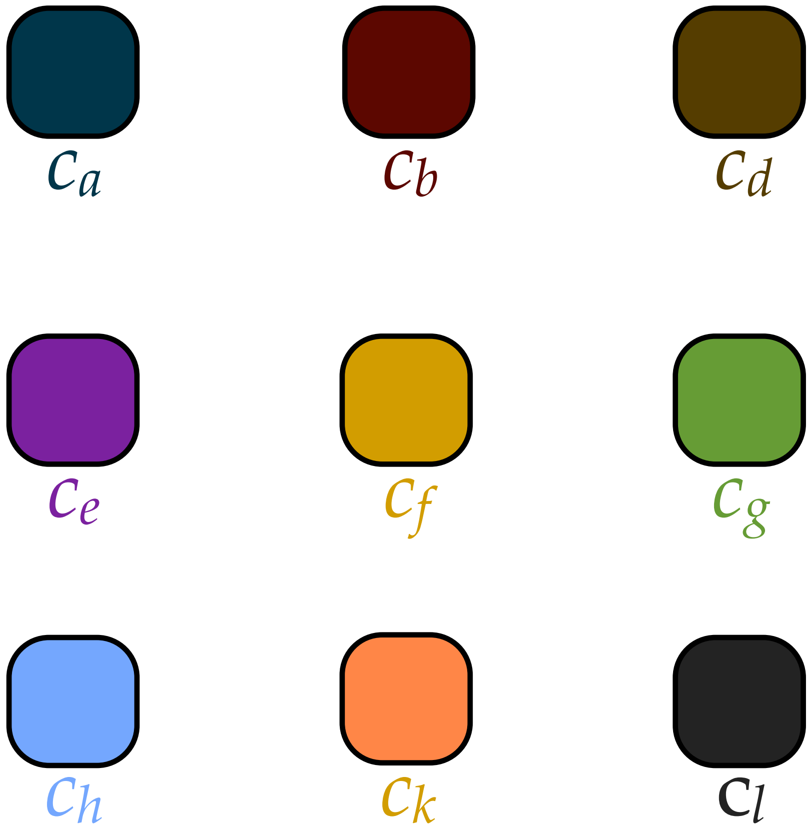


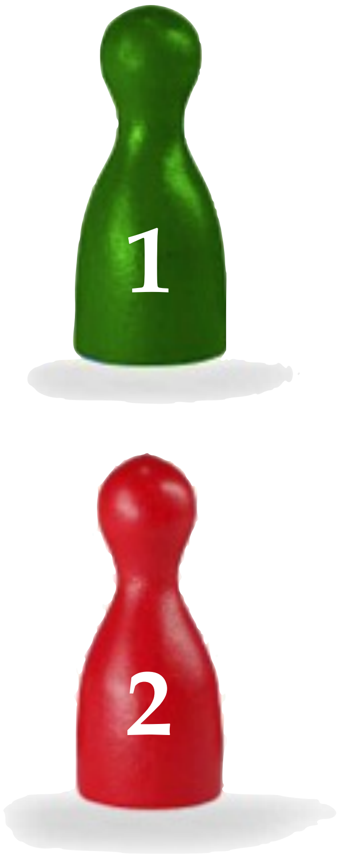
set of resources R



set of players N

Congestion games



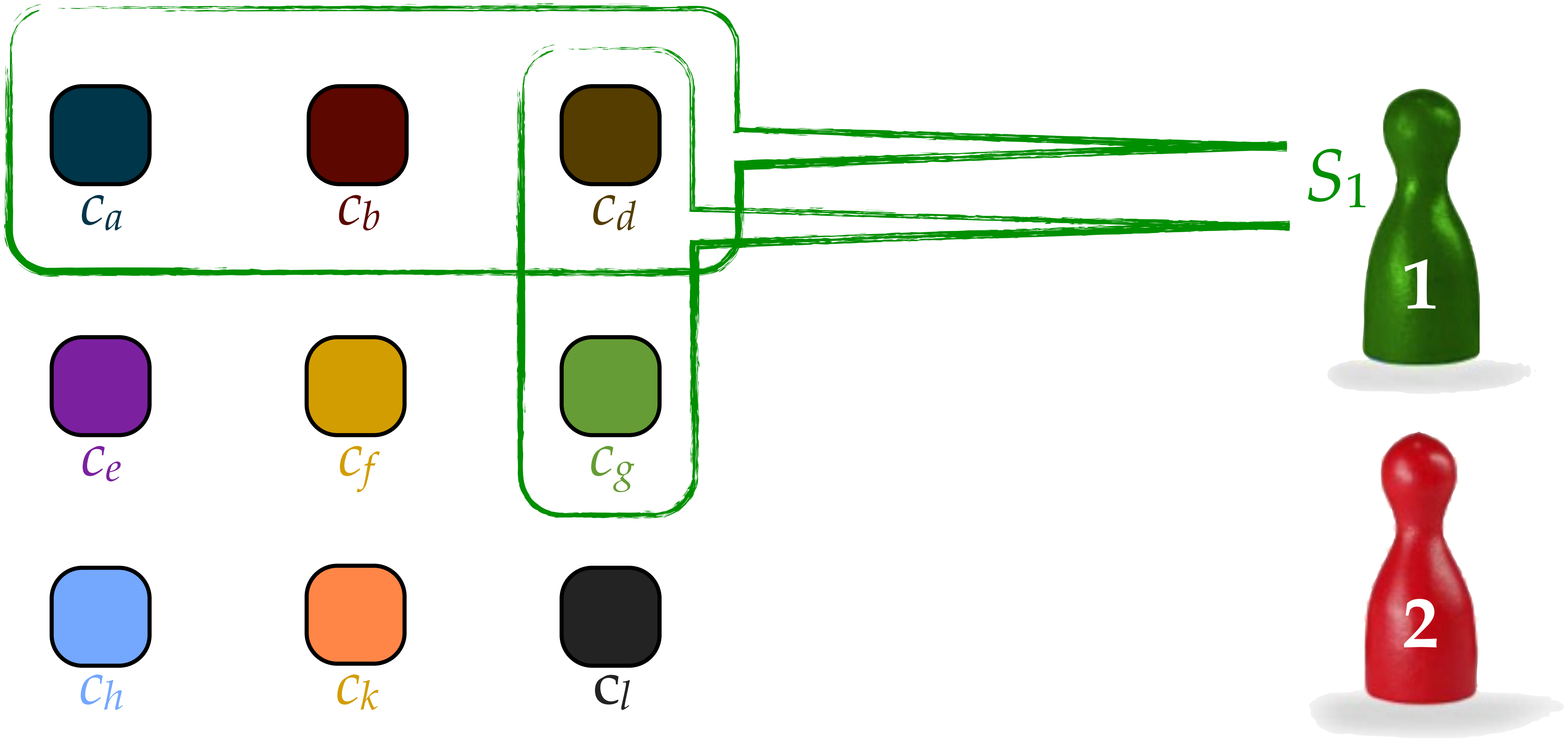


set of resources R

with cost functions $c_r : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$

set of players N

Congestion games



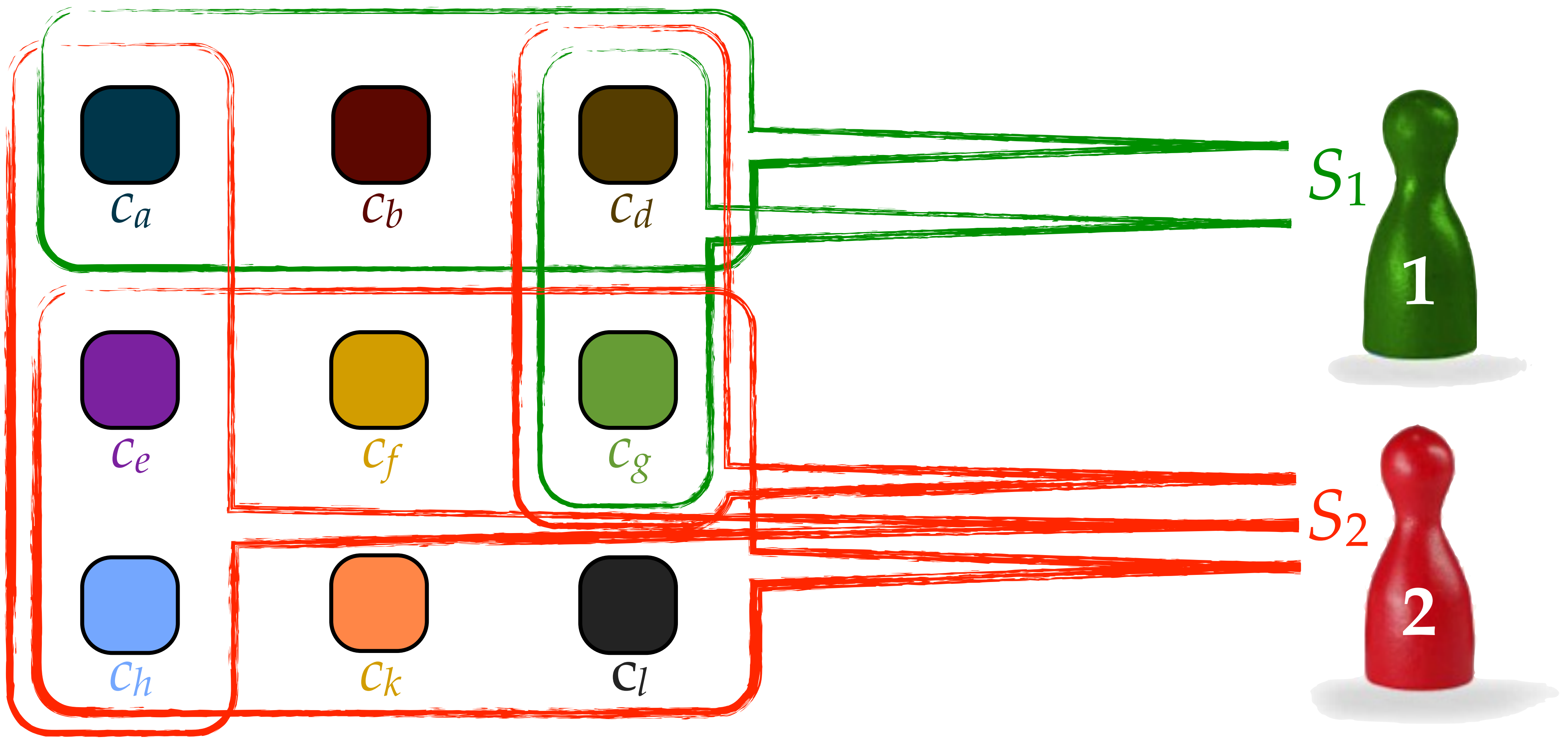
set of resources R

with cost functions $c_r : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$

set of players N

with strategies $S_i \subseteq 2^R$

Congestion games



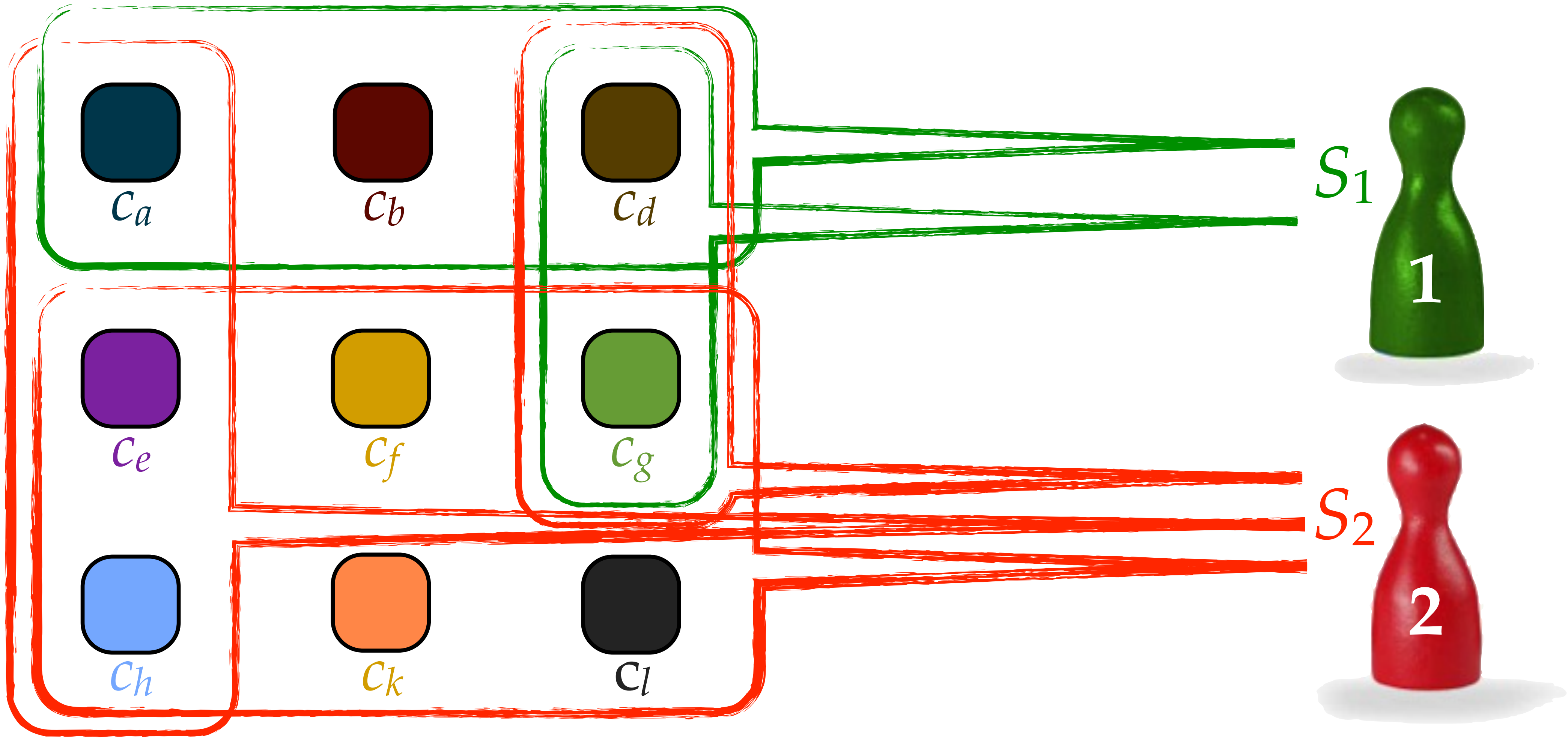
set of resources R

with cost functions $c_r : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$

set of players N

with strategies $S_i \subseteq 2^R$

Congestion games



set of resources R

with cost functions $c_r : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$

set of players N

with strategies $S_i \subseteq 2^R$

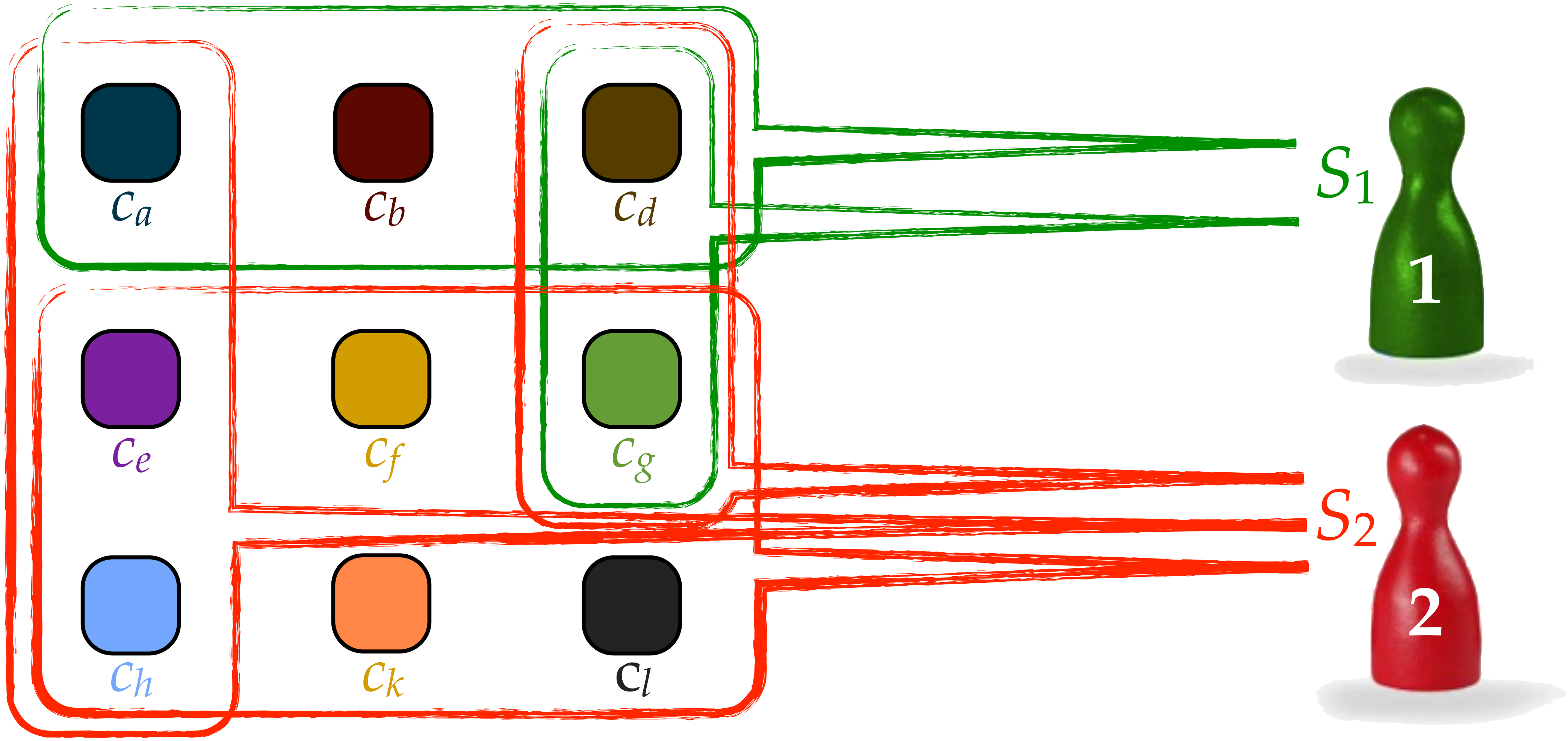
congestion game private costs: $\pi_i(s) = \sum_{r \in s_i} c_r(|\, j \in N : r \in s_j |\,)$

Congestion games

[Rosenthal, IJGT `73]

Theorem Congestion games have a Nash equilibrium.

Weighted congestion games



Set of Resources R

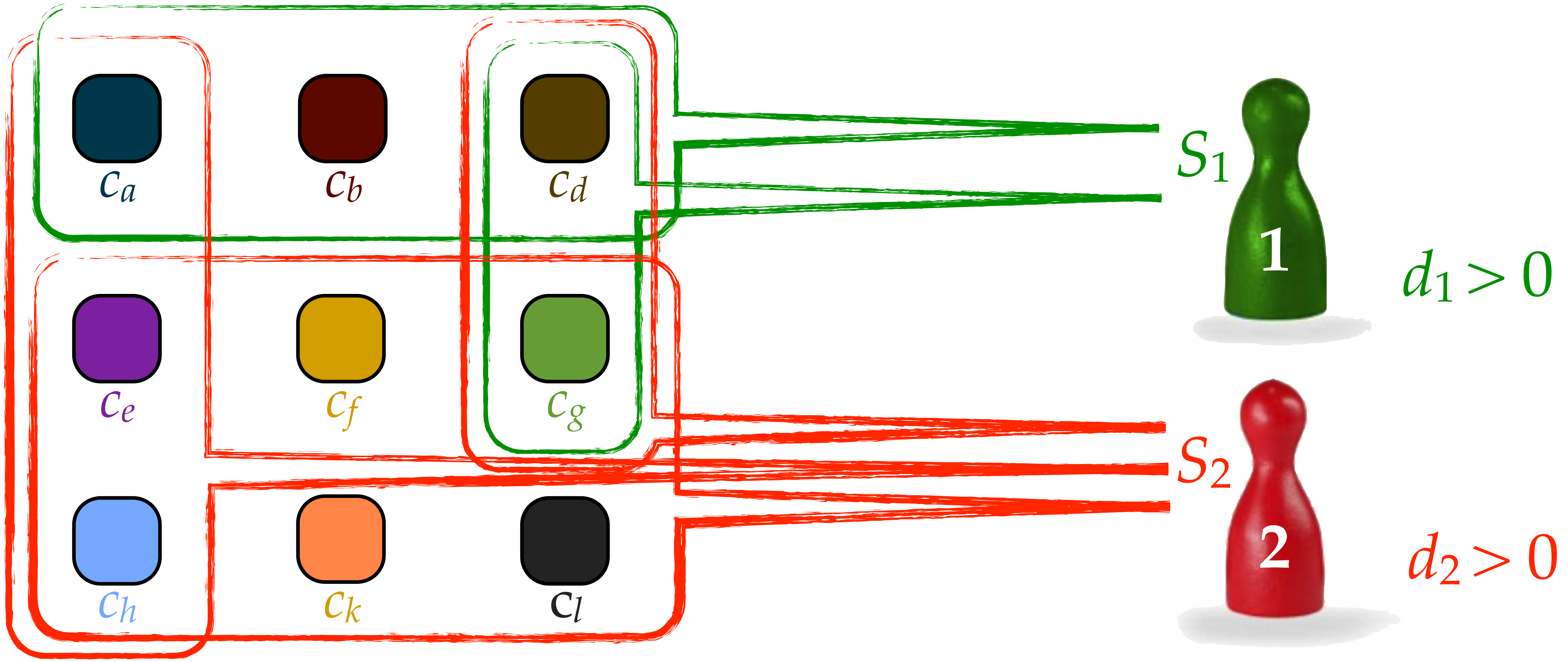
with cost functions $c_r : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$

Set of players N

with strategies $S_i \subseteq 2^R$

congestion game private costs: $\pi_i(s) = \sum_{r \in s_i} c_r(|\, j \in N : r \in s_j |\,)$

Weighted congestion games



Set of Resources R

with cost functions $c_r : \mathbb{R}_{\geq 0} \to \mathbb{R}$

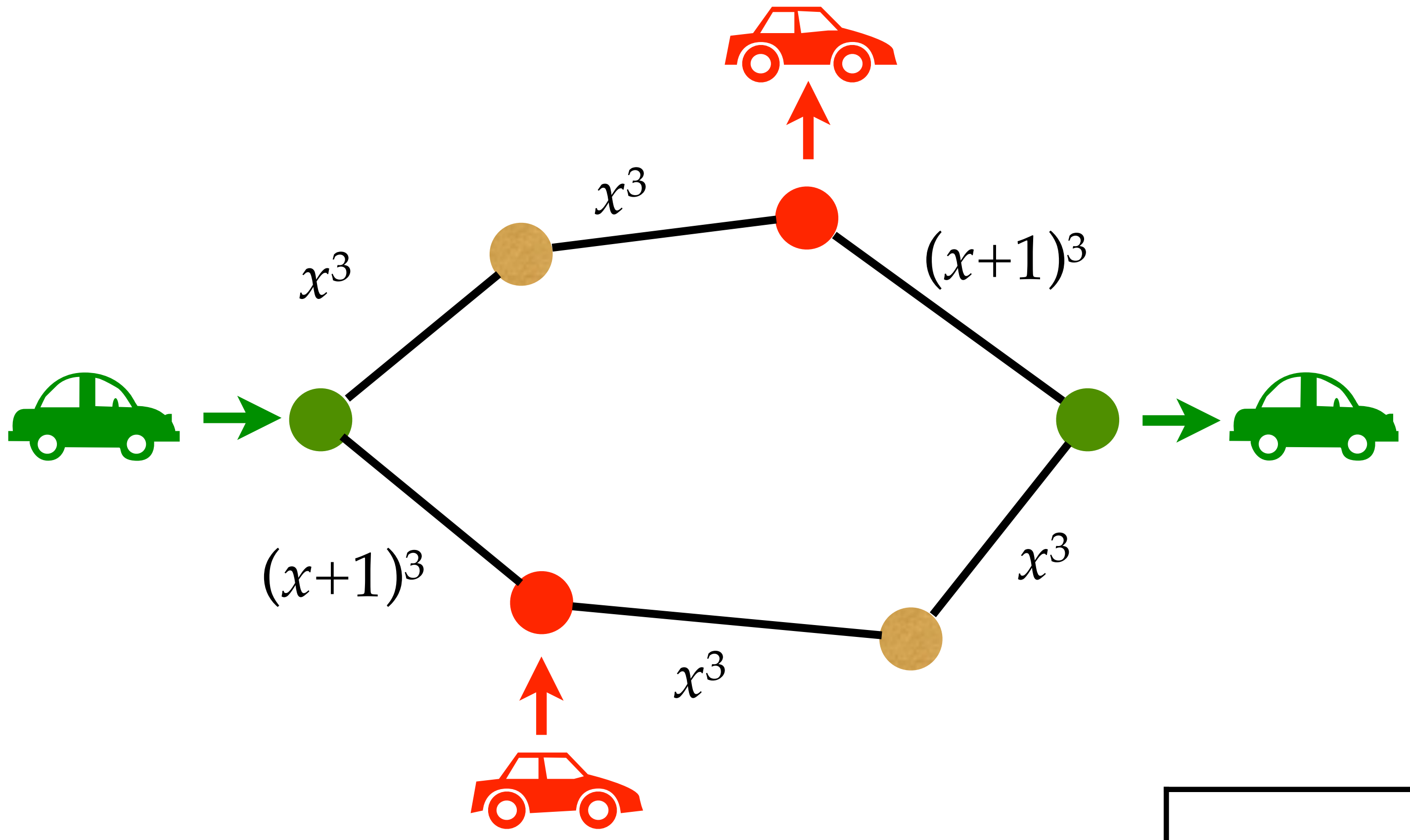
Set of players N

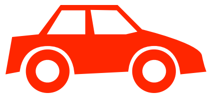
with strategies $S_i \subseteq 2^R$

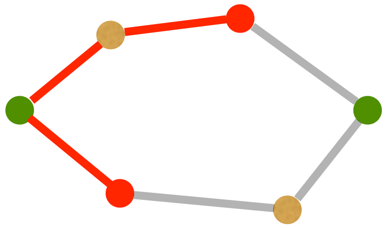
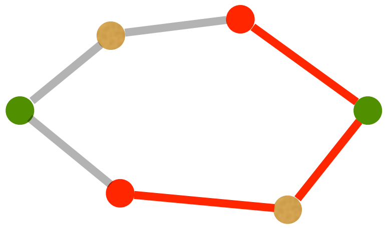
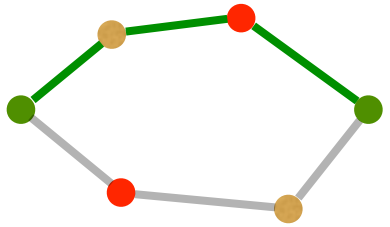
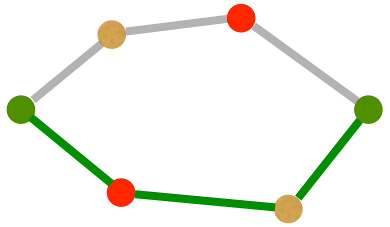
congestion game private costs: $\pi_i(s) = \sum_{r \in s_i} c_r(|\, j \in N : r \in s_j |\,)$

weighted congestion game private costs: $\pi_i(s) = \sum_{r \in s_i} d_i\, c_r(\sum_{j \in N\, :\, r \in s_j} d_j)$

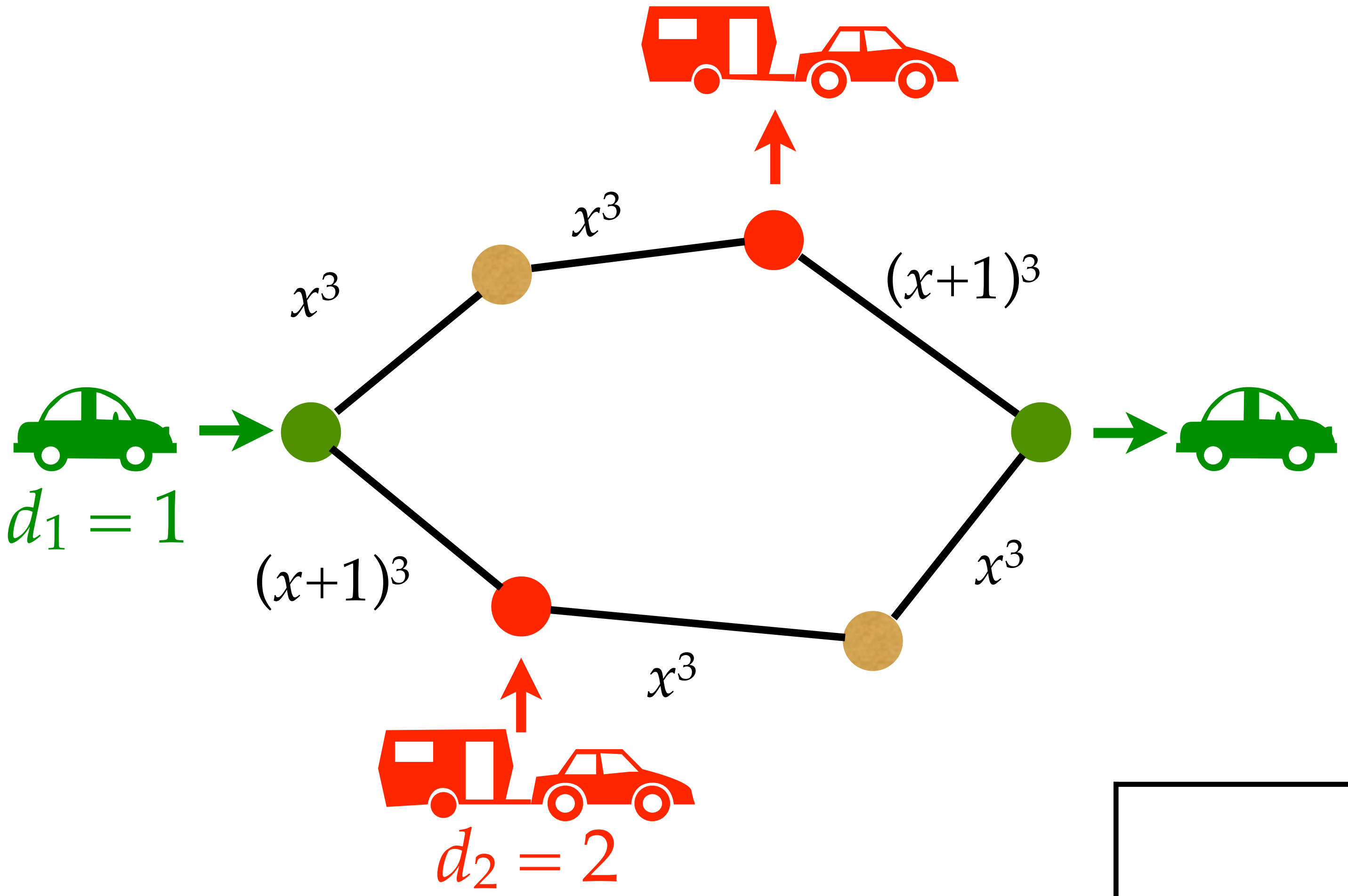
Weighted congestion games

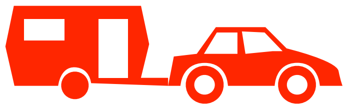


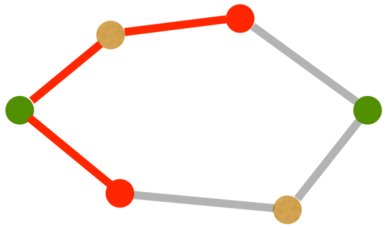
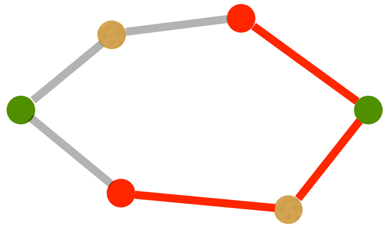
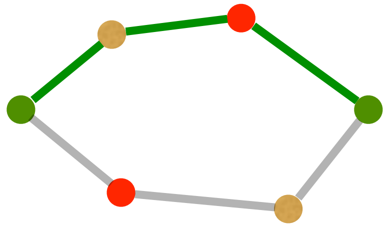
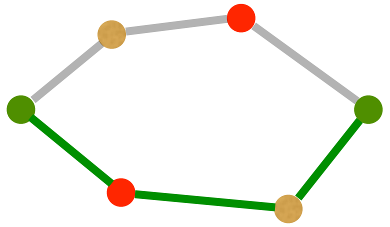


		
	24, 24	29, 29
	29, 29	24, 24

Weighted congestion games

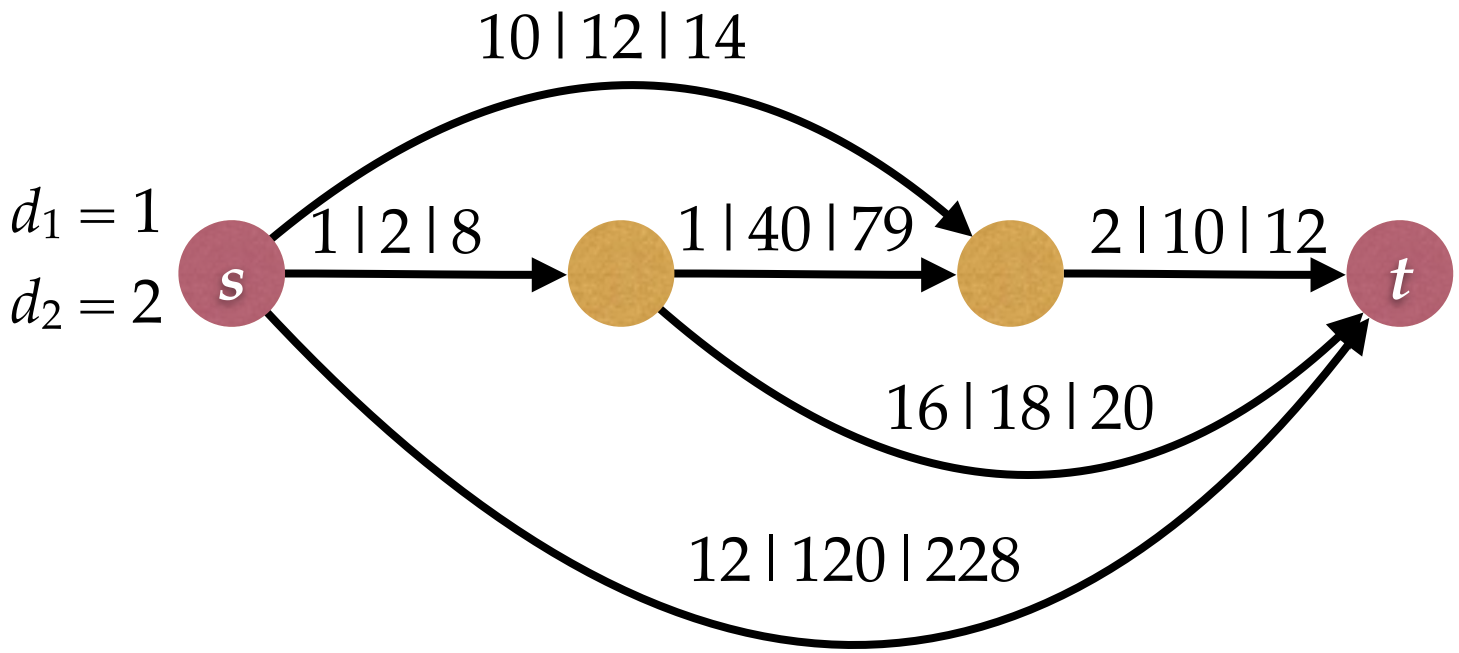




		
	62, 162	66, 160
	66, 160	62, 162

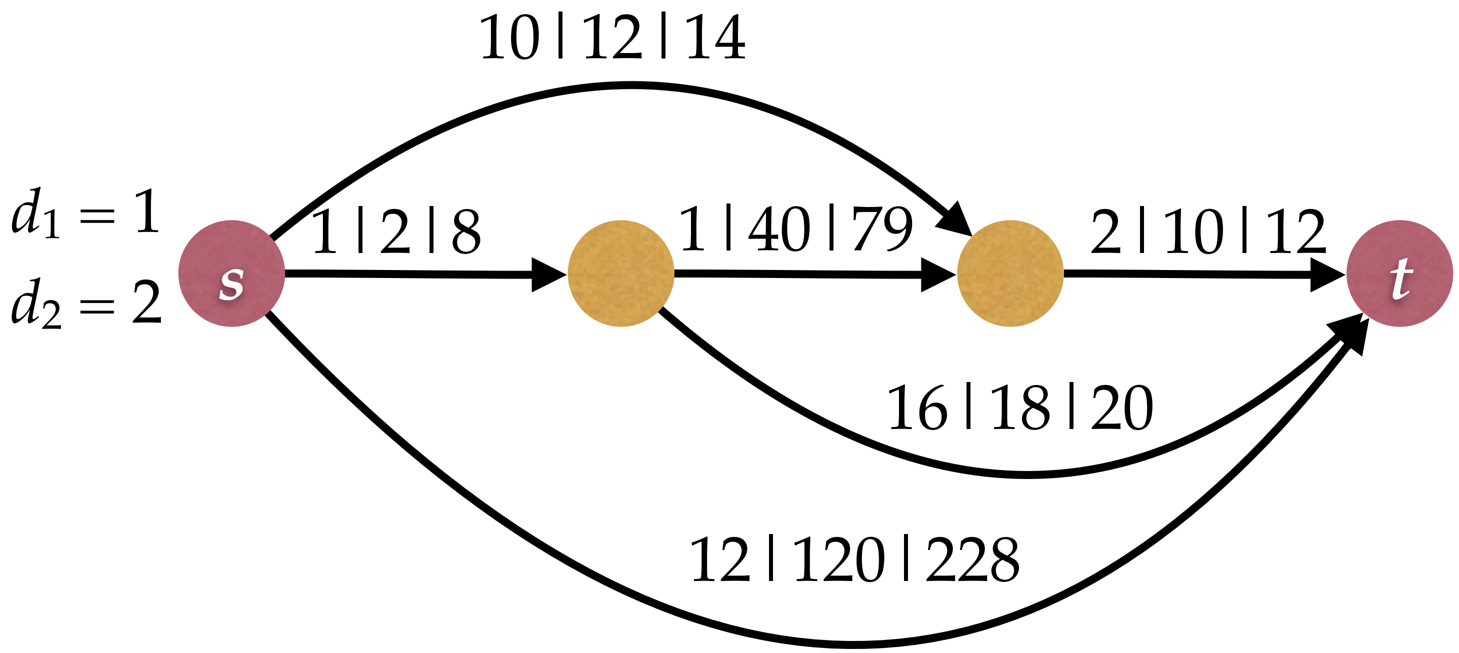
Further counterexamples

[Fotakis et al.,TCS `05]

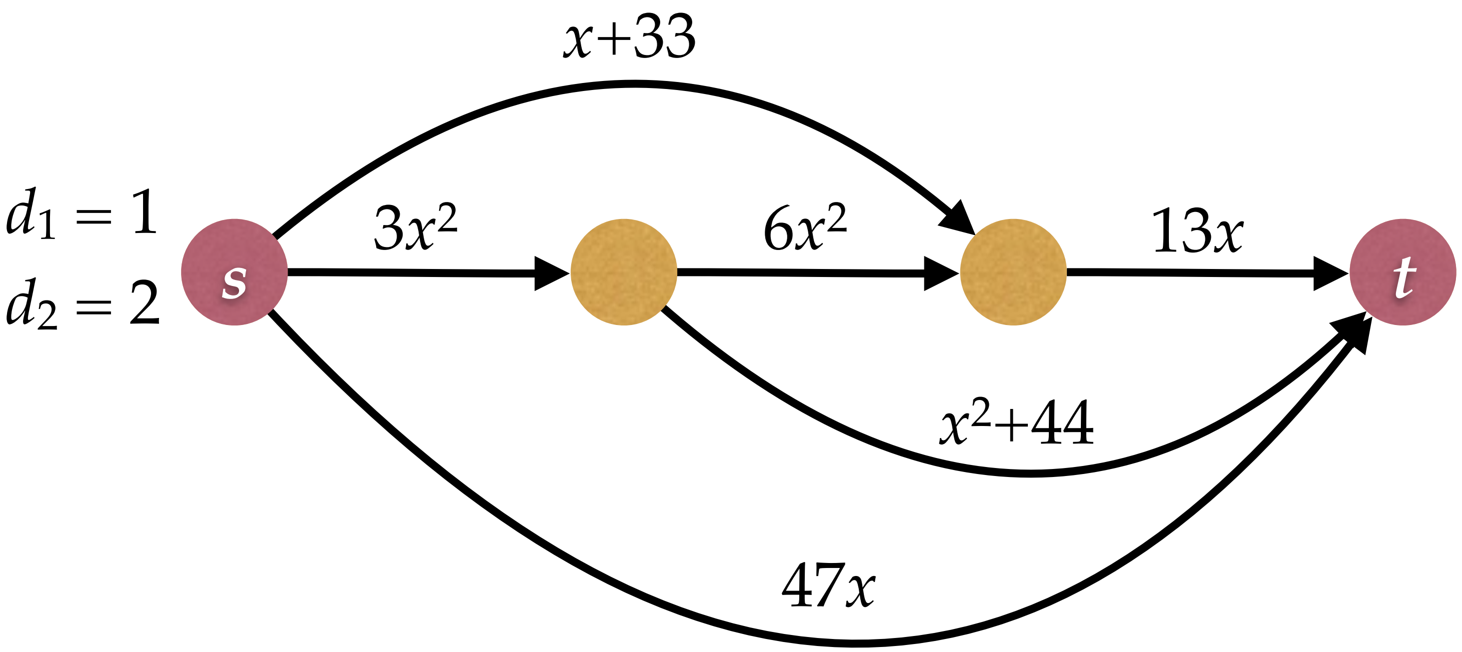


Further counterexamples

[Fotakis et al.,TCS `05]

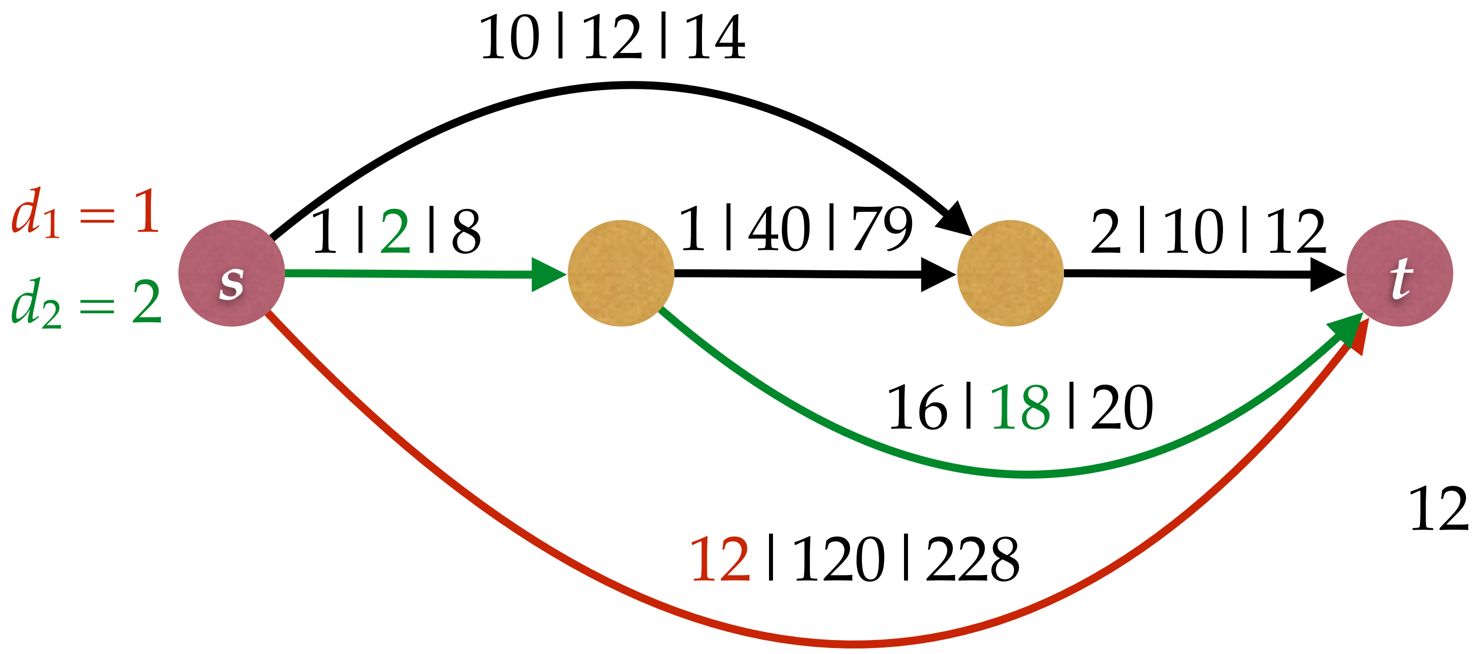


[Goemans et al., FOCS `05]



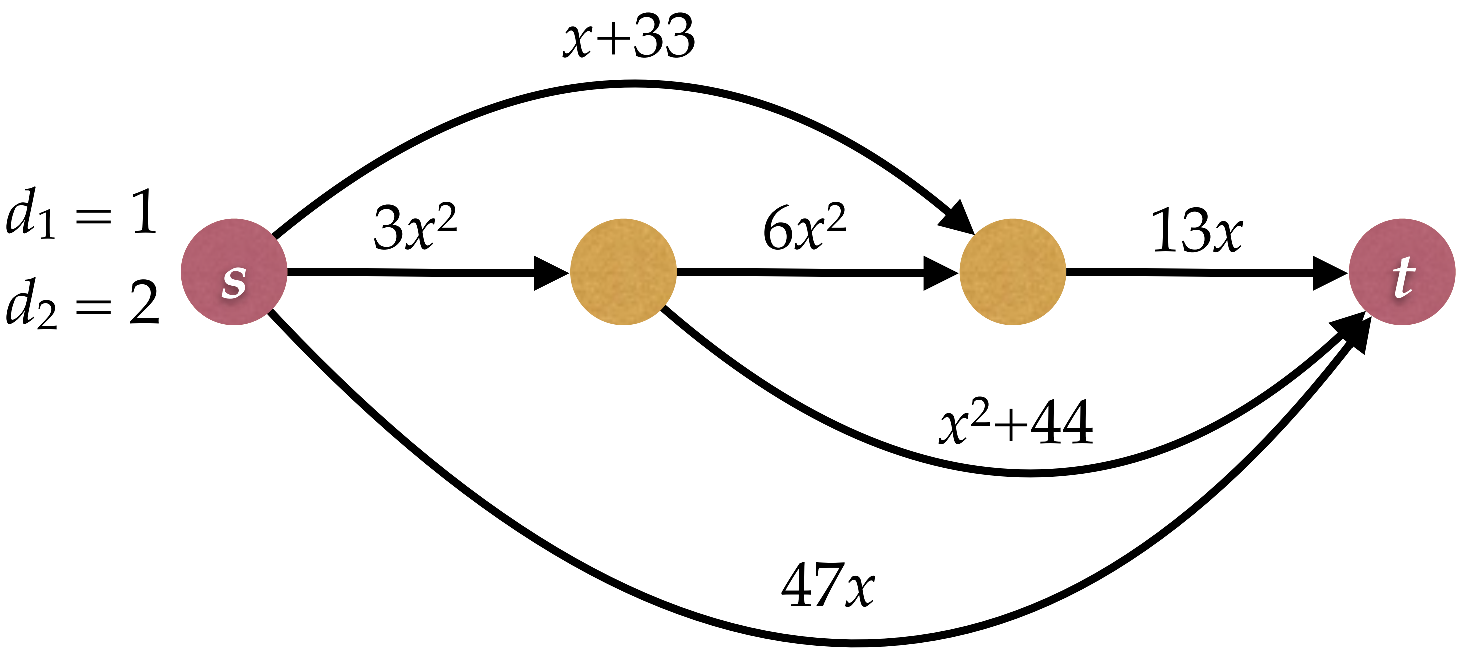
Further counterexamples

[Fotakis et al., TCS `05]



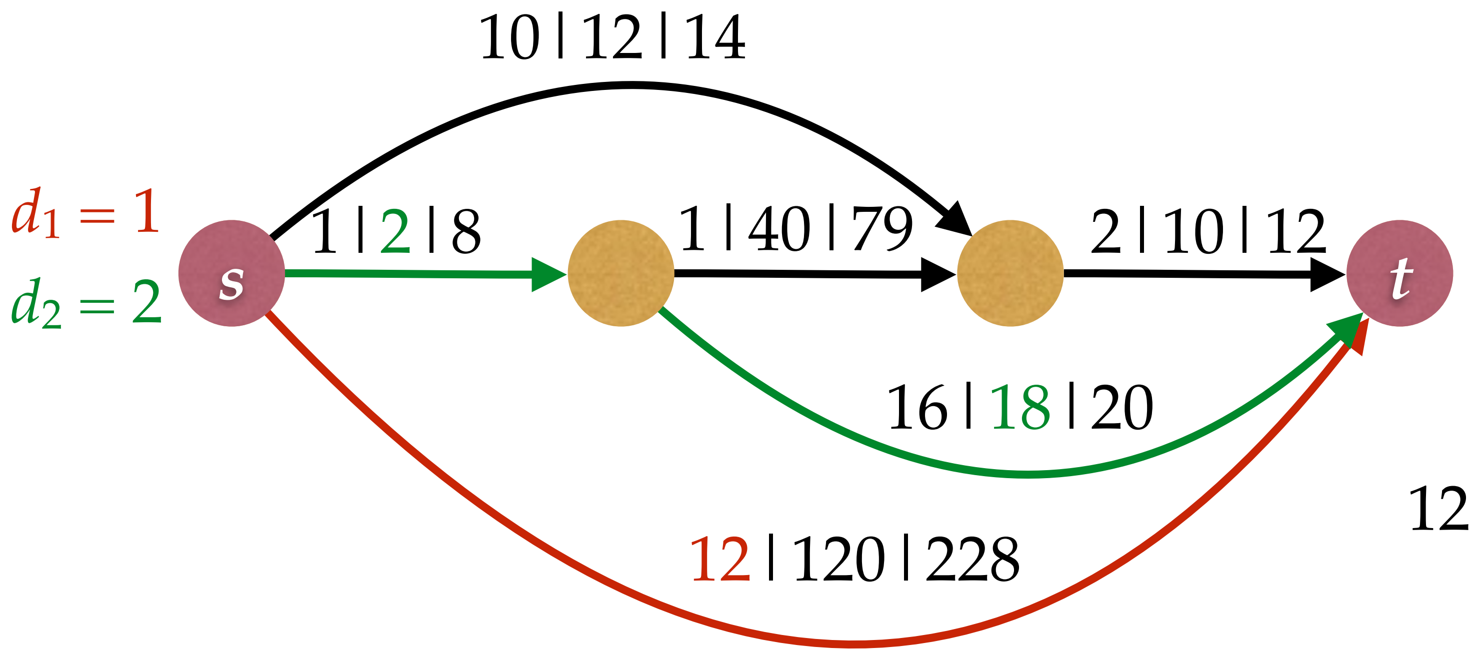
12/11-approximate equilibrium

[Goemans et al., FOCS `05]



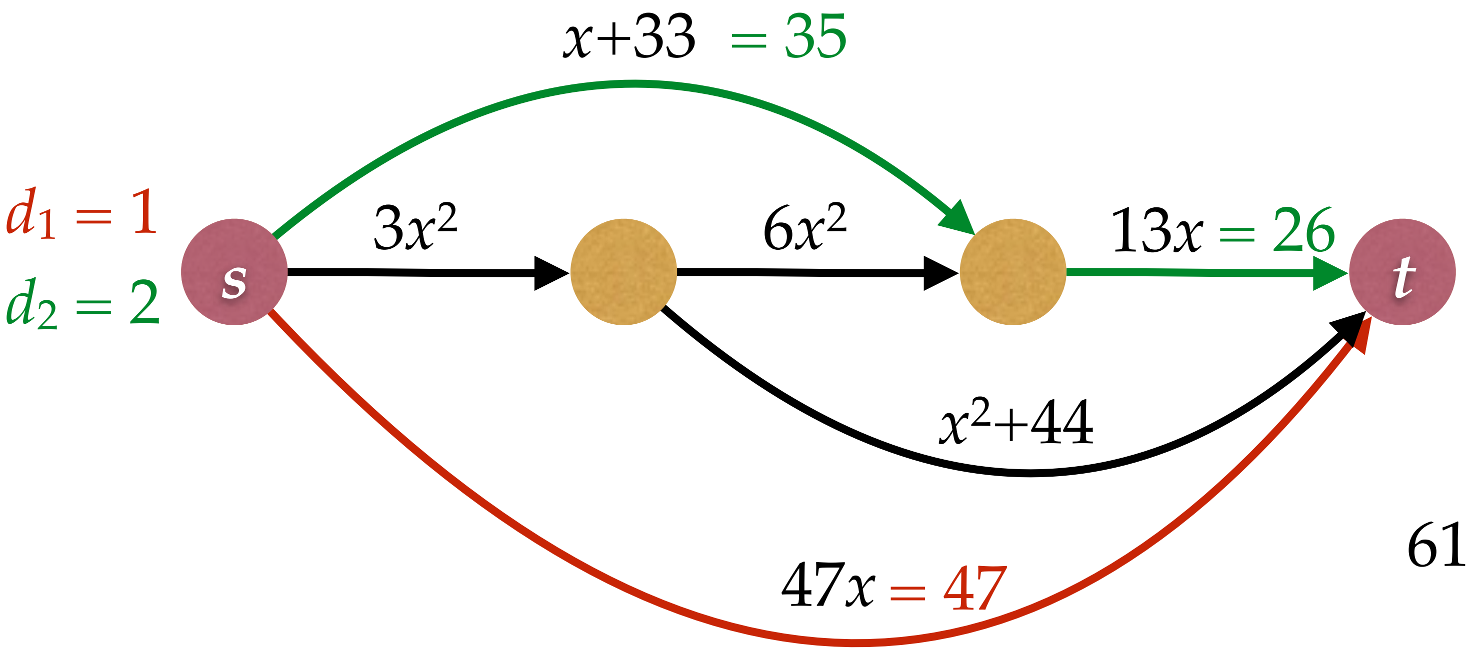
Further counterexamples

[Fotakis et al., TCS `05]



$12/11$-approximate equilibrium

[Goemans et al., FOCS `05]



$61/60$-approximate equilibrium

Previous work

[Caragiannis et al.,EC `12]

Functions	Approximation factors
quadratic	2
cubic	6
polynomials of max. degree Δ	$\Delta!$

Our results

Functions	Approximation factors
quadratic	2
cubic	6
polynomials of max. degree Δ	$\Delta+1$

Our results

Functions	Approximation factors
quadratic	$4/3$
cubic	$1.785\ldots$
polynomials of max. degree Δ	$\Delta+1$

Our results

Functions	Approximation factors
quadratic	$4/3$
cubic	$1.785...$
polynomials of max. degree Δ	$\Delta+1$
concave	$3/2$

Our results

Functions	Approximation factors	
	2 players	all games
quadratic	1.054...	4/3
cubic	1.074...	1.785...
polynomials of max. degree Δ		Δ+1
concave		3/2

The potential function argument

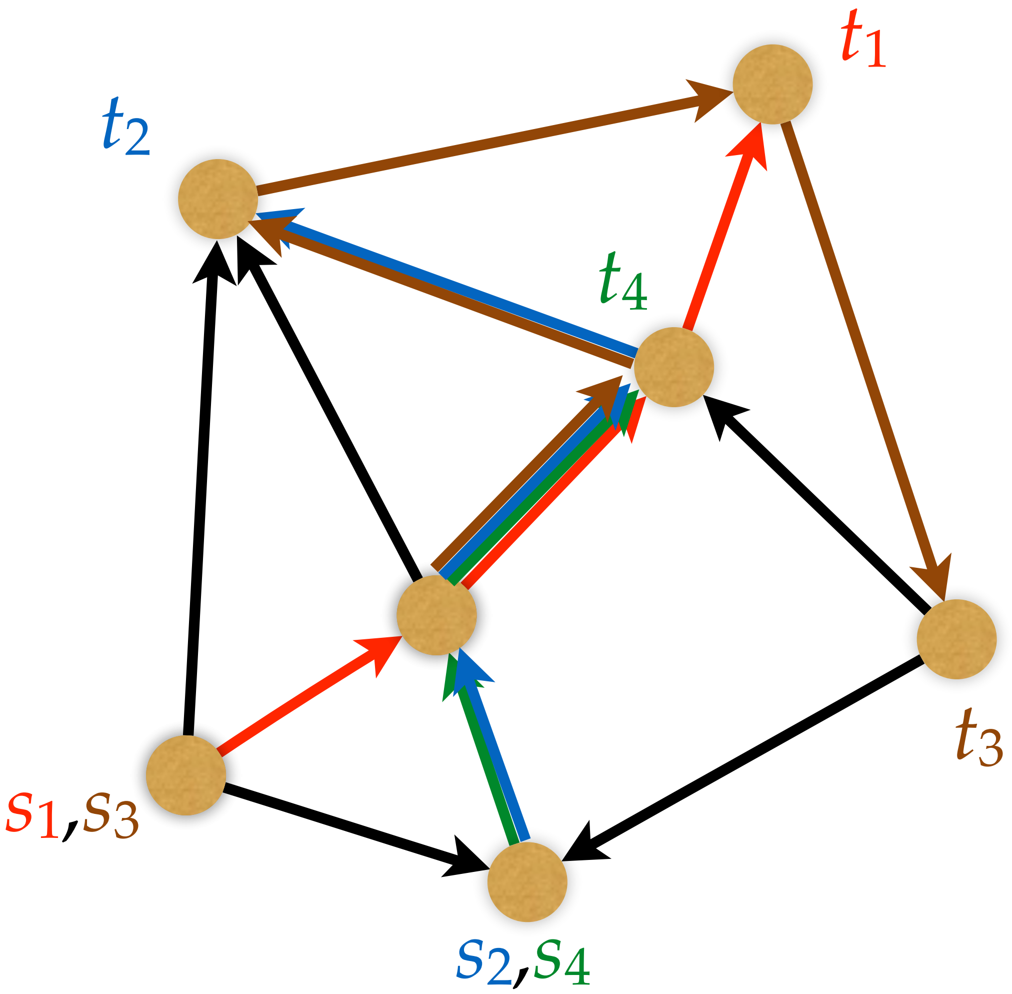
[Rosenthal, IJGT `73]

- Let $P(s) = \sum_{r \in R} P_r(s)$, where $\sum_{k=1,\ldots,d_r(s)} c_r(k)$

The potential function argument

[Rosenthal, IJGT `73]

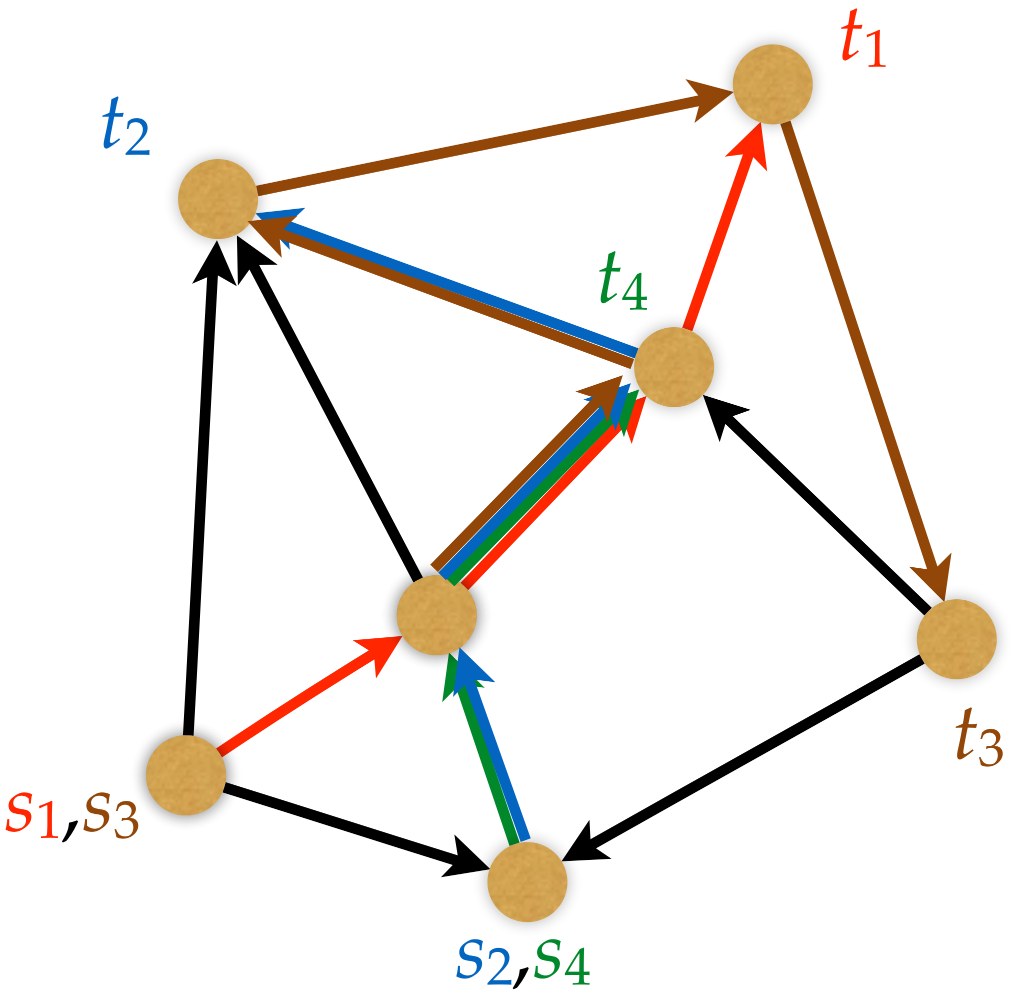
- Let $P(s) = \sum_{r \in R} P_r(s)$, where $\sum_{k=1,\ldots,d_r(s)} c_r(k)$

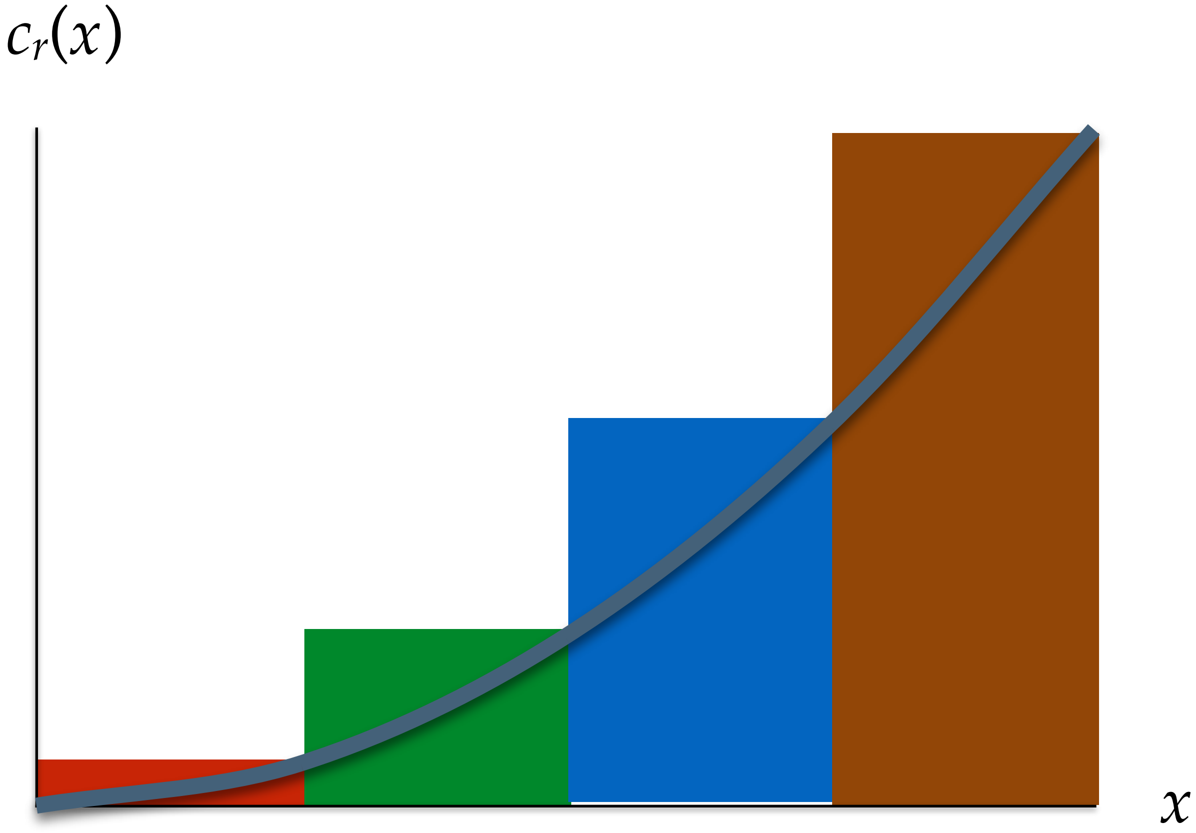


The potential function argument

[Rosenthal, IJGT `73]

- Let $P(s) = \sum_{r\in R} P_r(s)$, where $\sum_{k=1,\ldots,d_r(s)} c_r(k)$

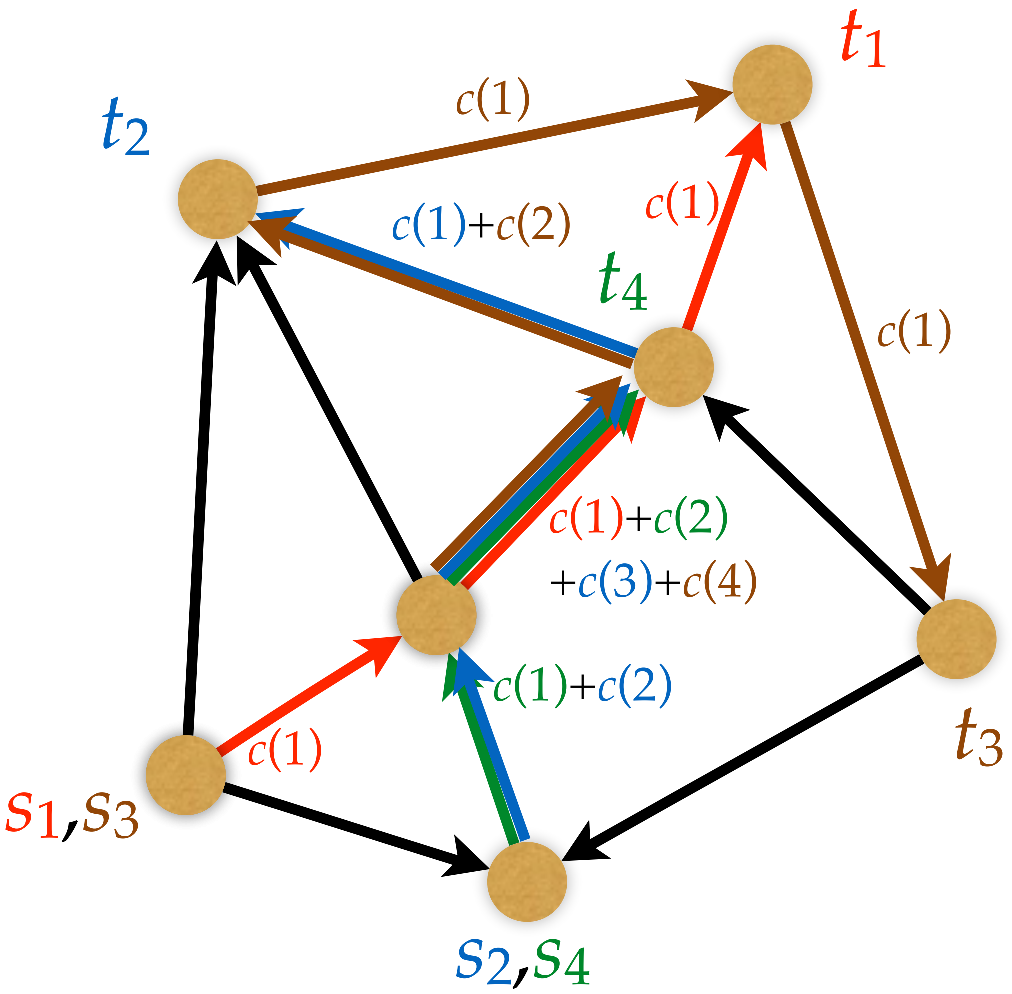


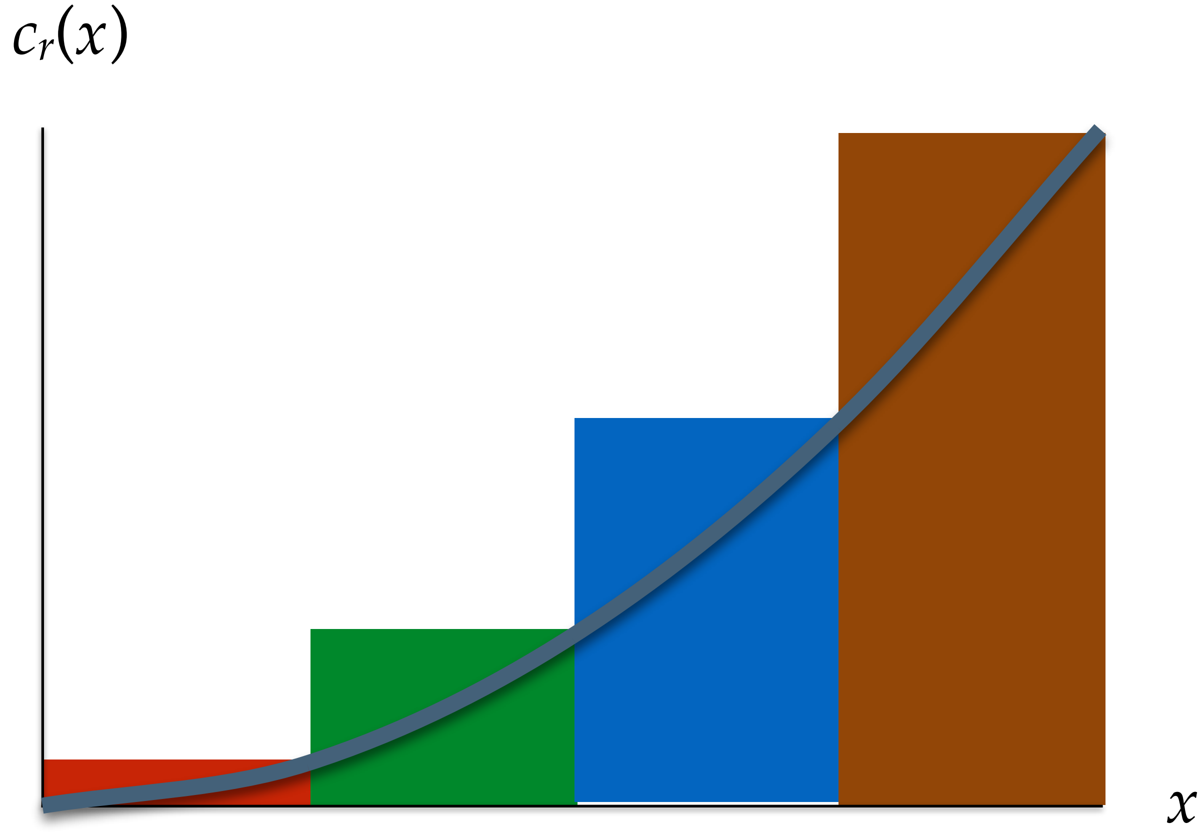


The potential function argument

[Rosenthal, IJGT `73]

- Let $P(s) = \sum_{r\in R} P_r(s)$, where $\sum_{k=1,\ldots,d_r(s)} c_r(k)$

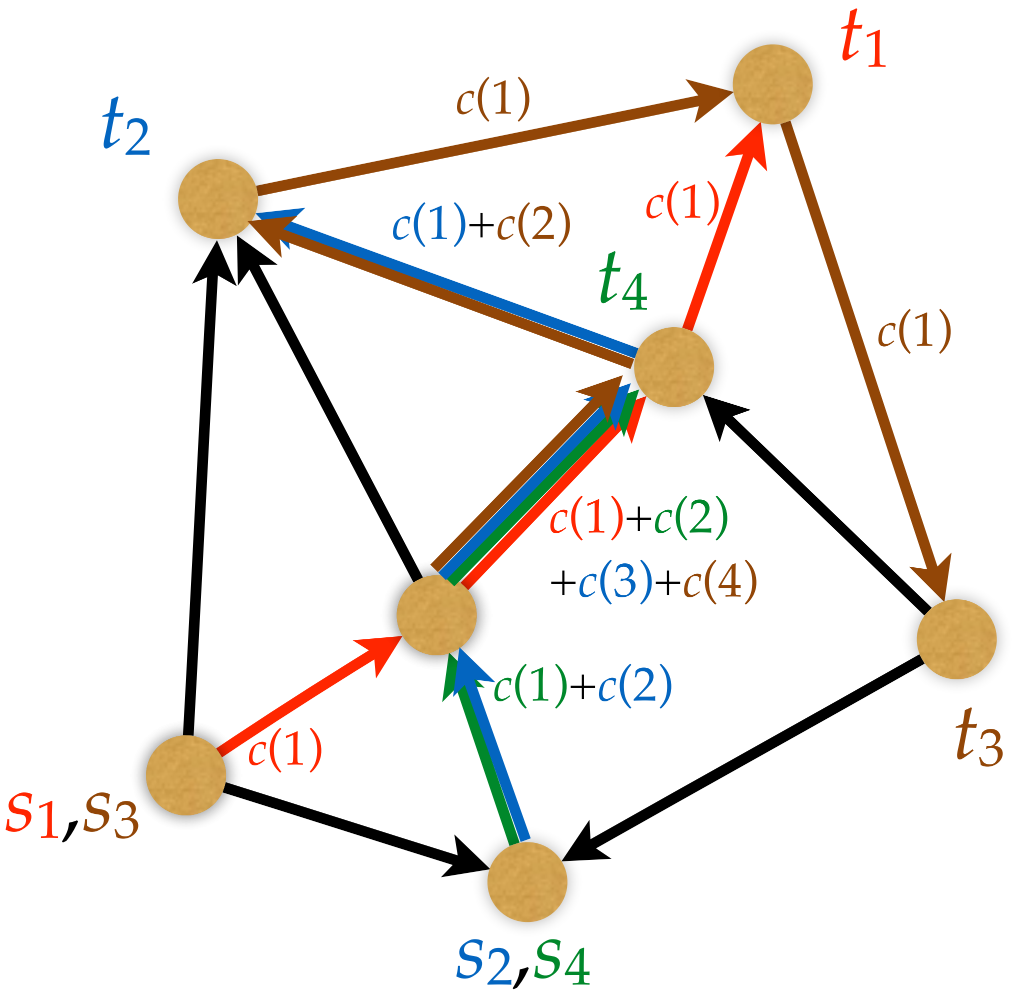


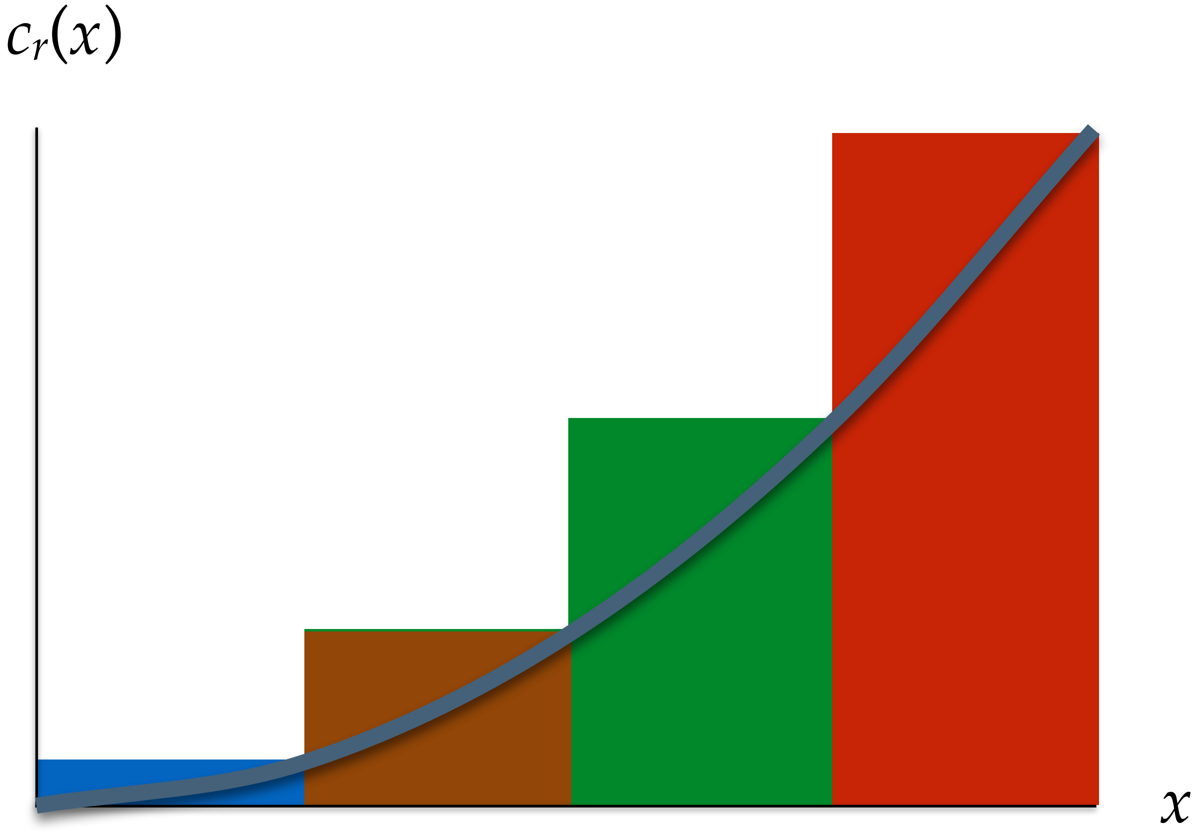


The potential function argument

[Rosenthal, IJGT `73]

- Let $P(s) = \sum_{r \in R} P_r(s)$, where $\sum_{k=1,\ldots,d_r(s)} c_r(k)$

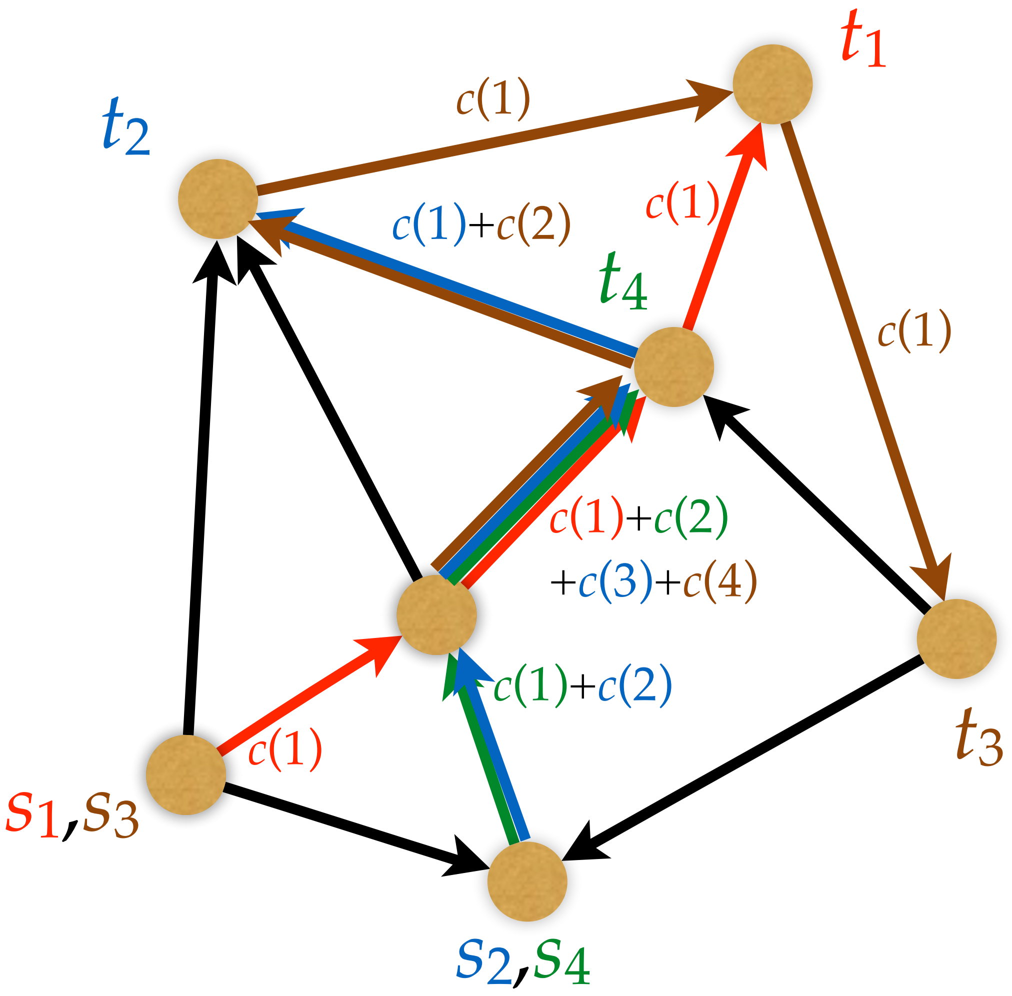


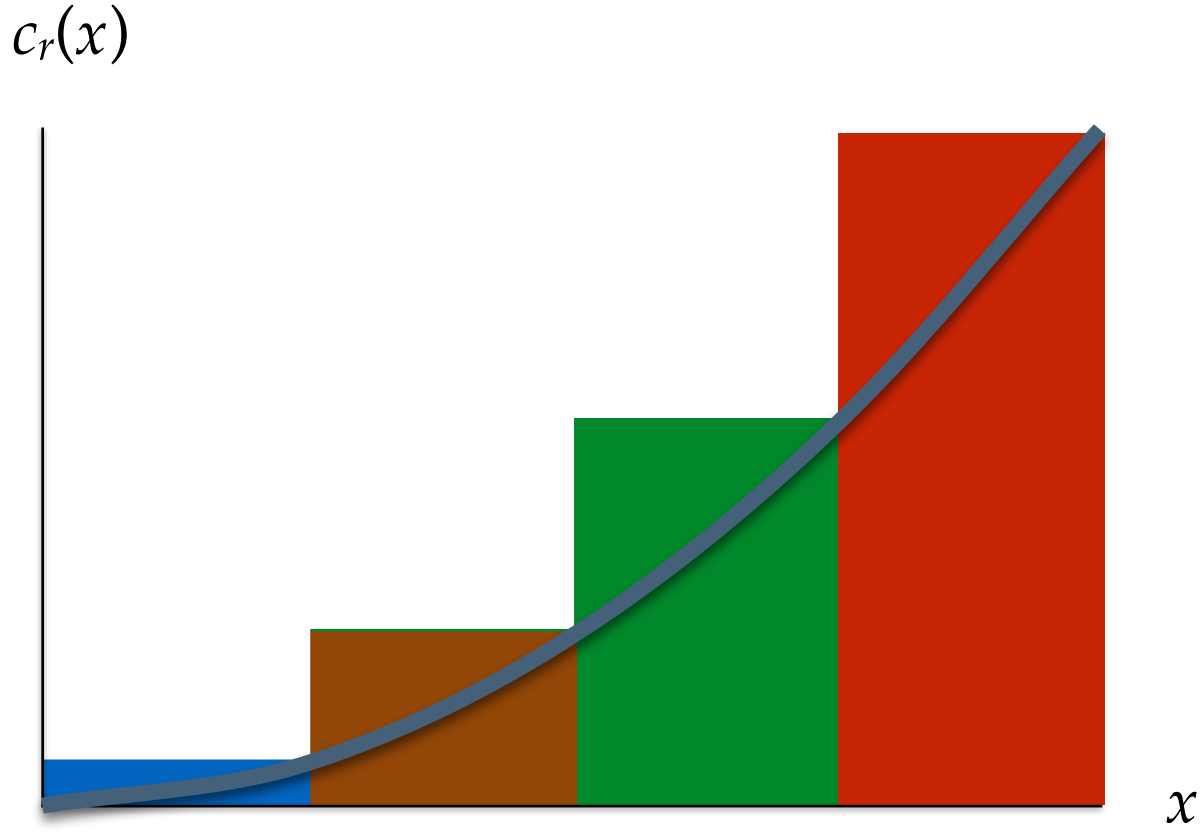


The potential function argument

[Rosenthal, IJGT `73]

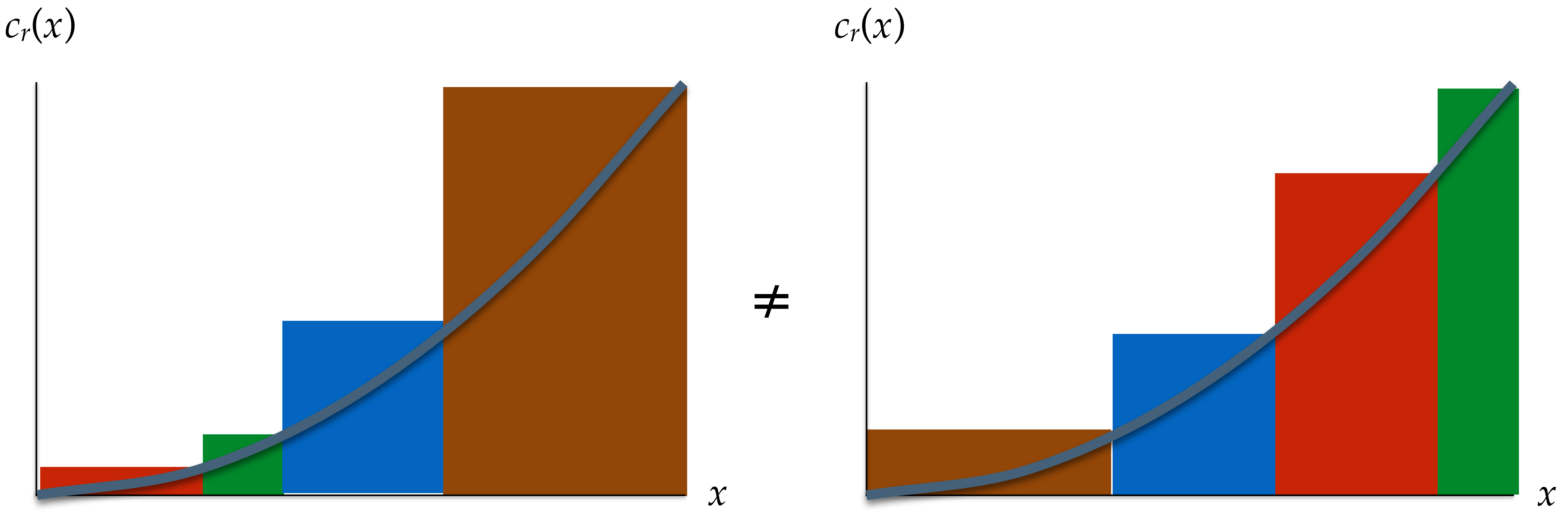
- Let $P(s) = \sum_{r\in R} P_r(s)$, where $\sum_{k=1,\dots,d_r(s)} c_r(k)$



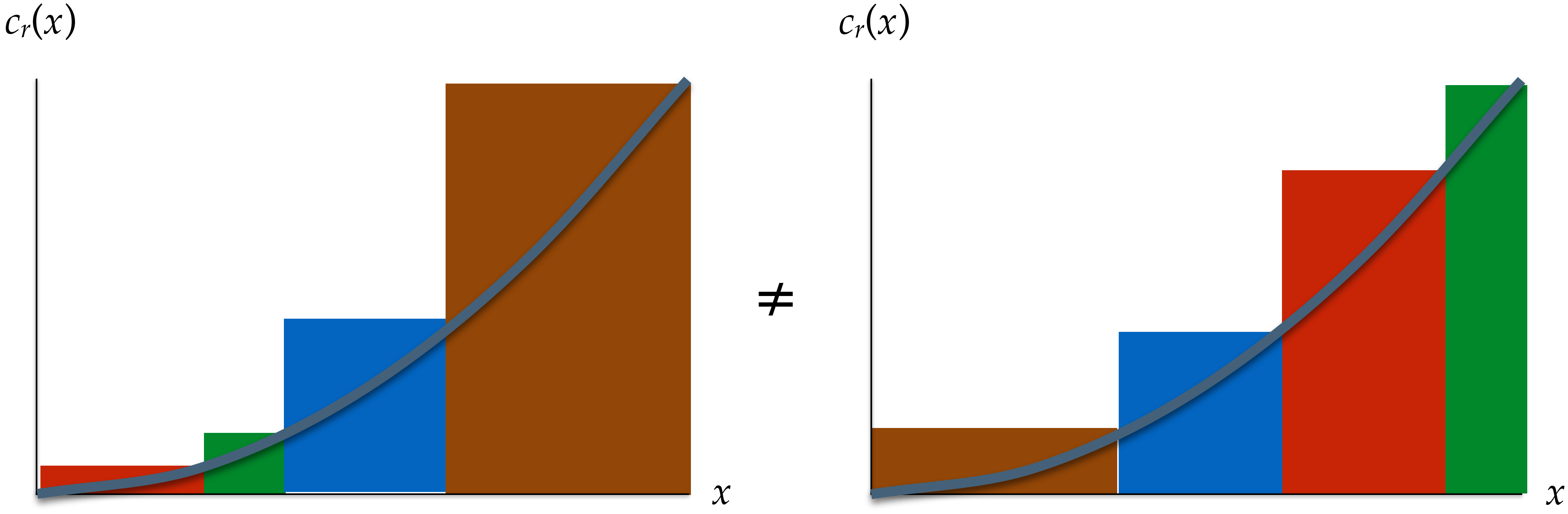


$$P(t_n,s_{-n}) - P(s) = \sum_{r\in t_n} c_r(d_r(t_n,s_{-n})) - \sum_{r\in s_n} c_r(d_r(s)) = \pi_i(t_i,s_{-i}) - \pi_i(s)$$

Weighted players



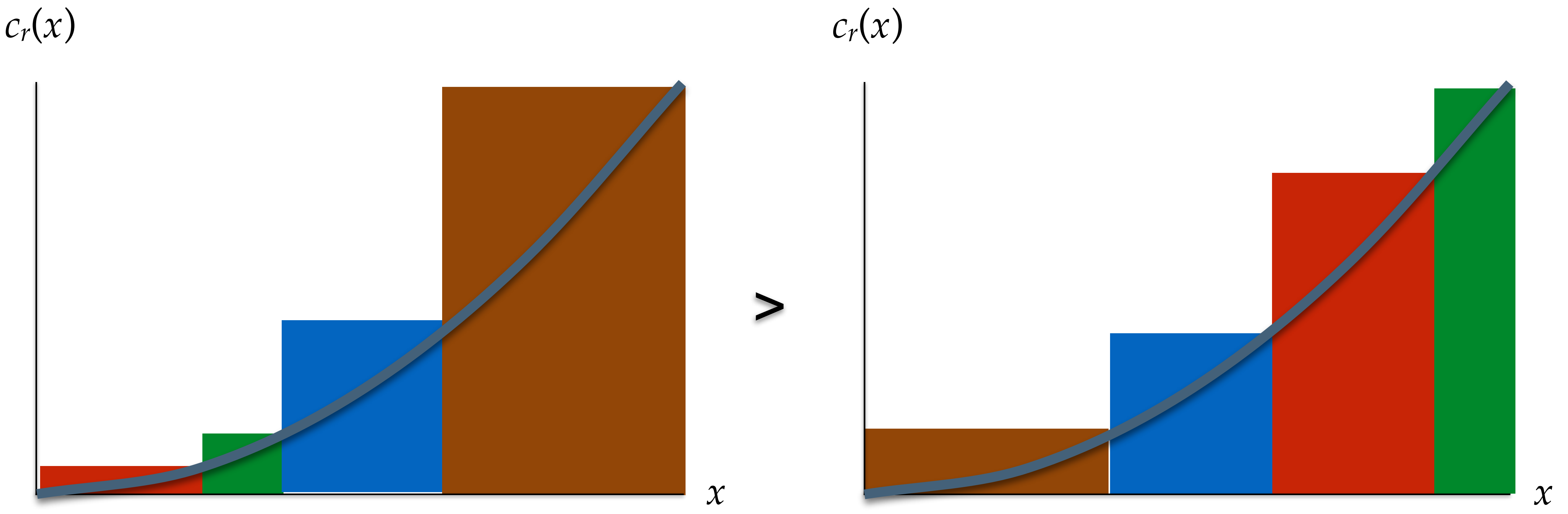
Weighted players



Observation For a convex function,

- the potential is minimized for non-increasing players
- the potential is maximized for non-decreasing players

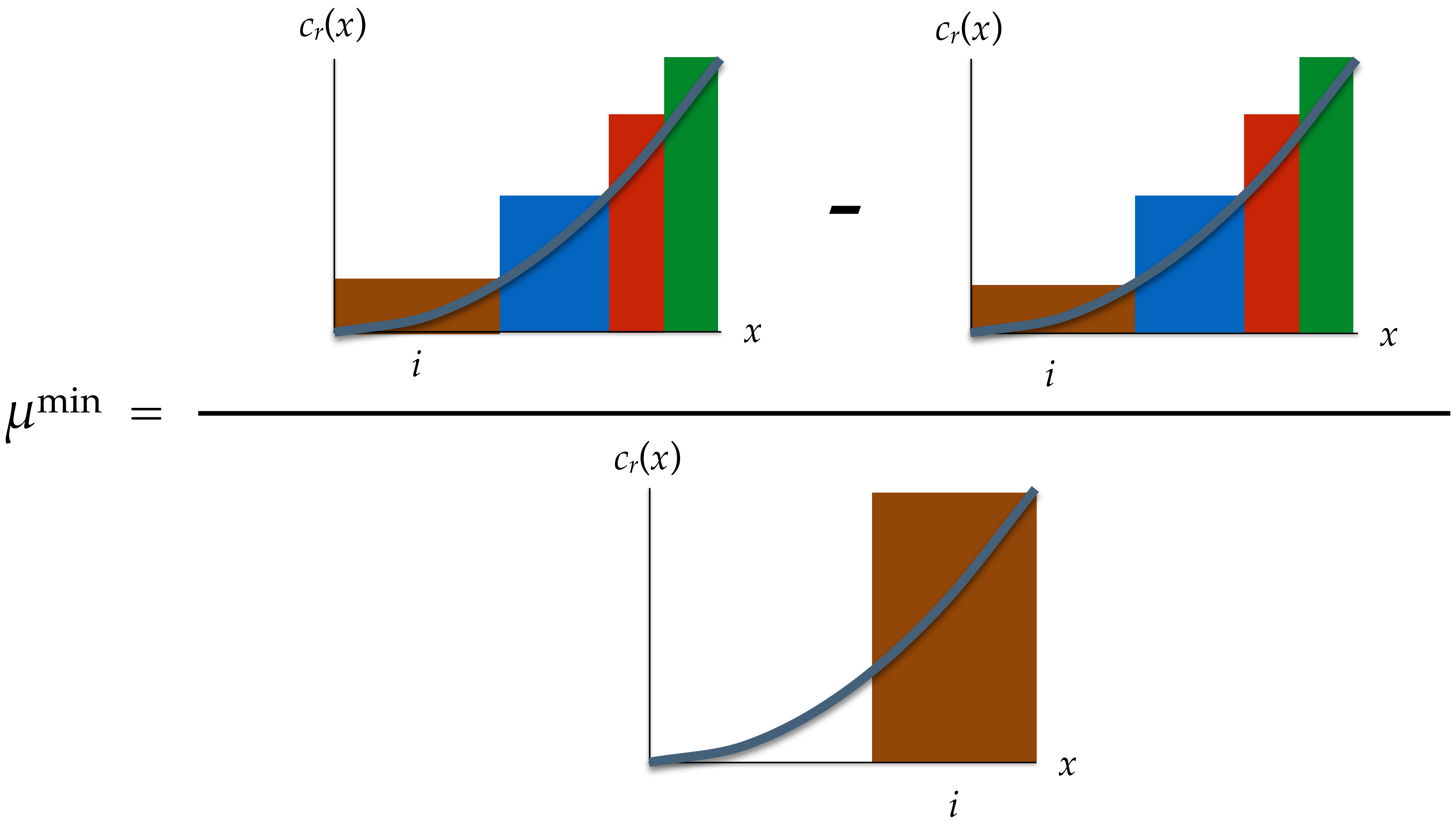
Weighted players



Observation For a convex function,

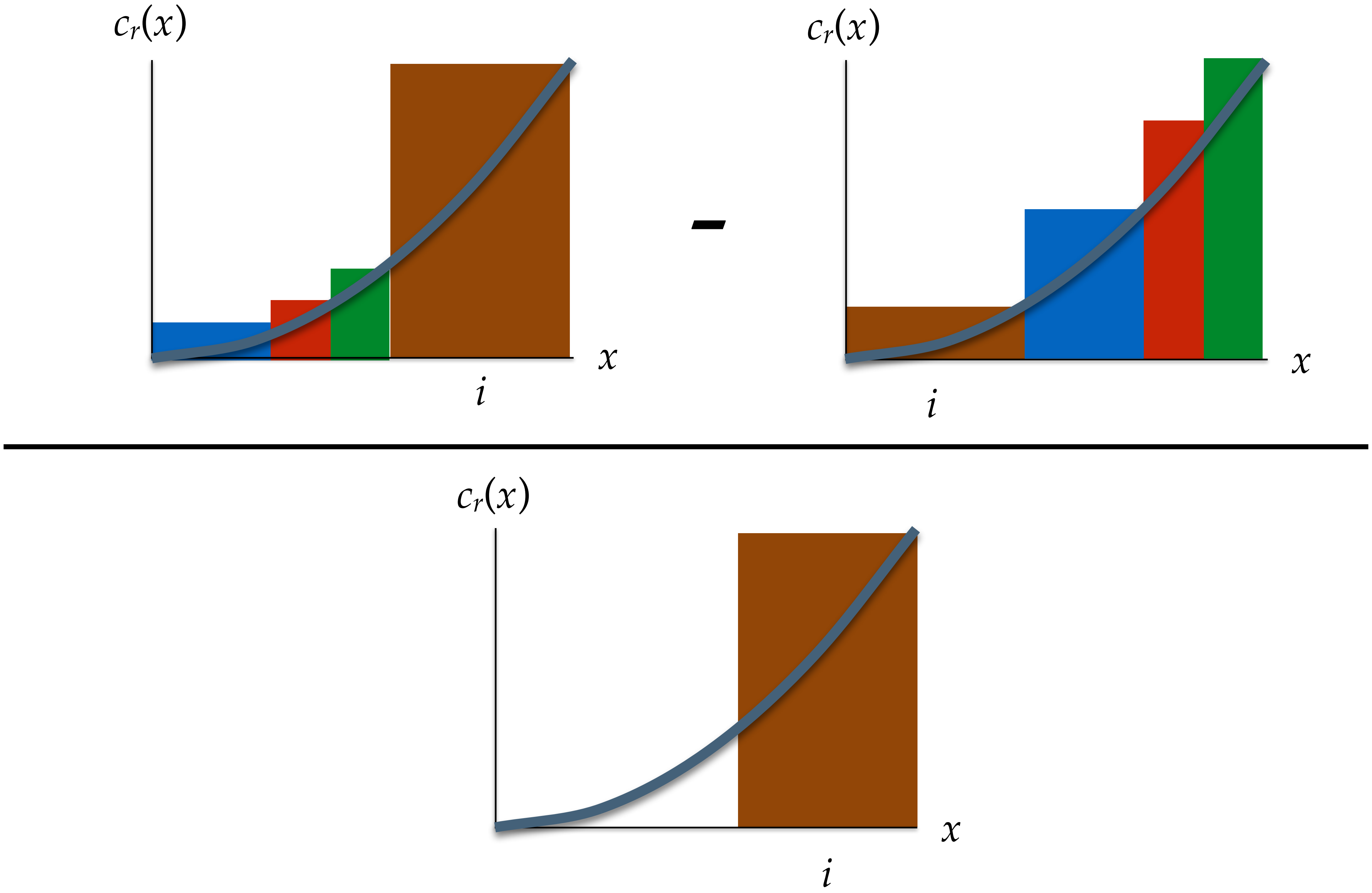
- the potential is minimized for non-increasing players
- the potential is maximized for non-decreasing players

Bounding the approximation factor

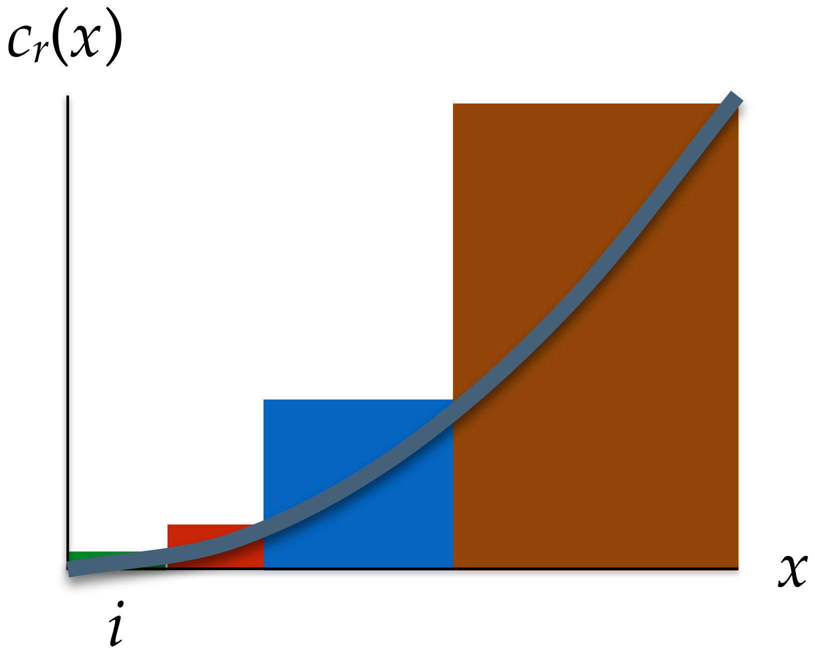


Bounding the approximation factor

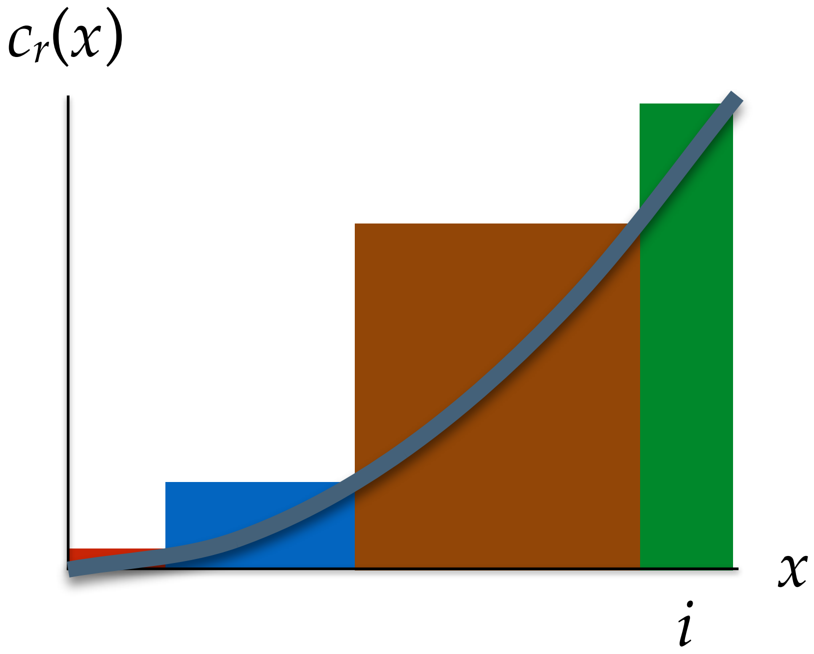
$\mu^{\min} =$



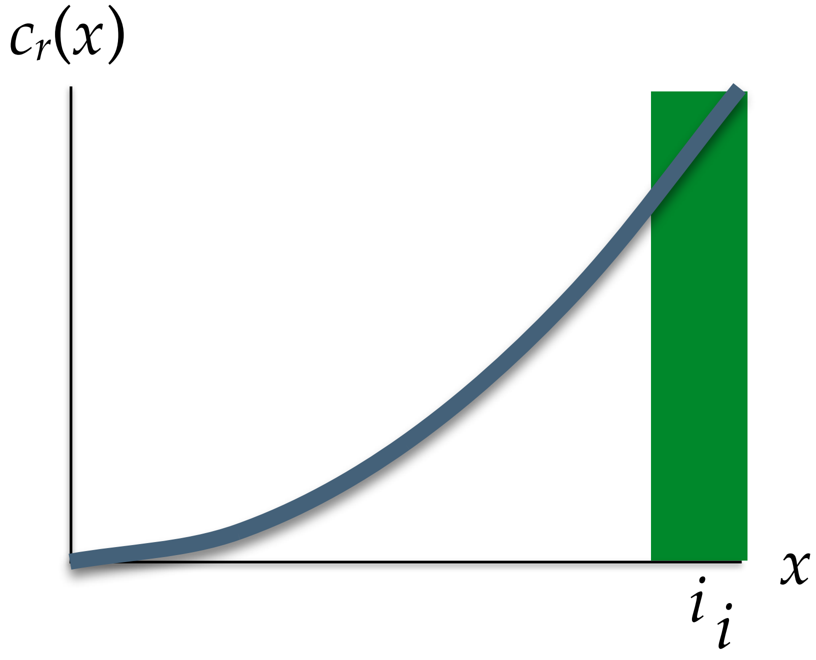
Bounding the approximation factor



-



$\mu^{\max} =$



Main result

[Hansknecht K. Skopalik, `14]

Theorem Weighted congestion game have an α-approximate pure Nash equilibrium, where $\alpha \leq \min \{1+\mu^{\max}, 1/(1-\mu^{\min})\}$.

Main result

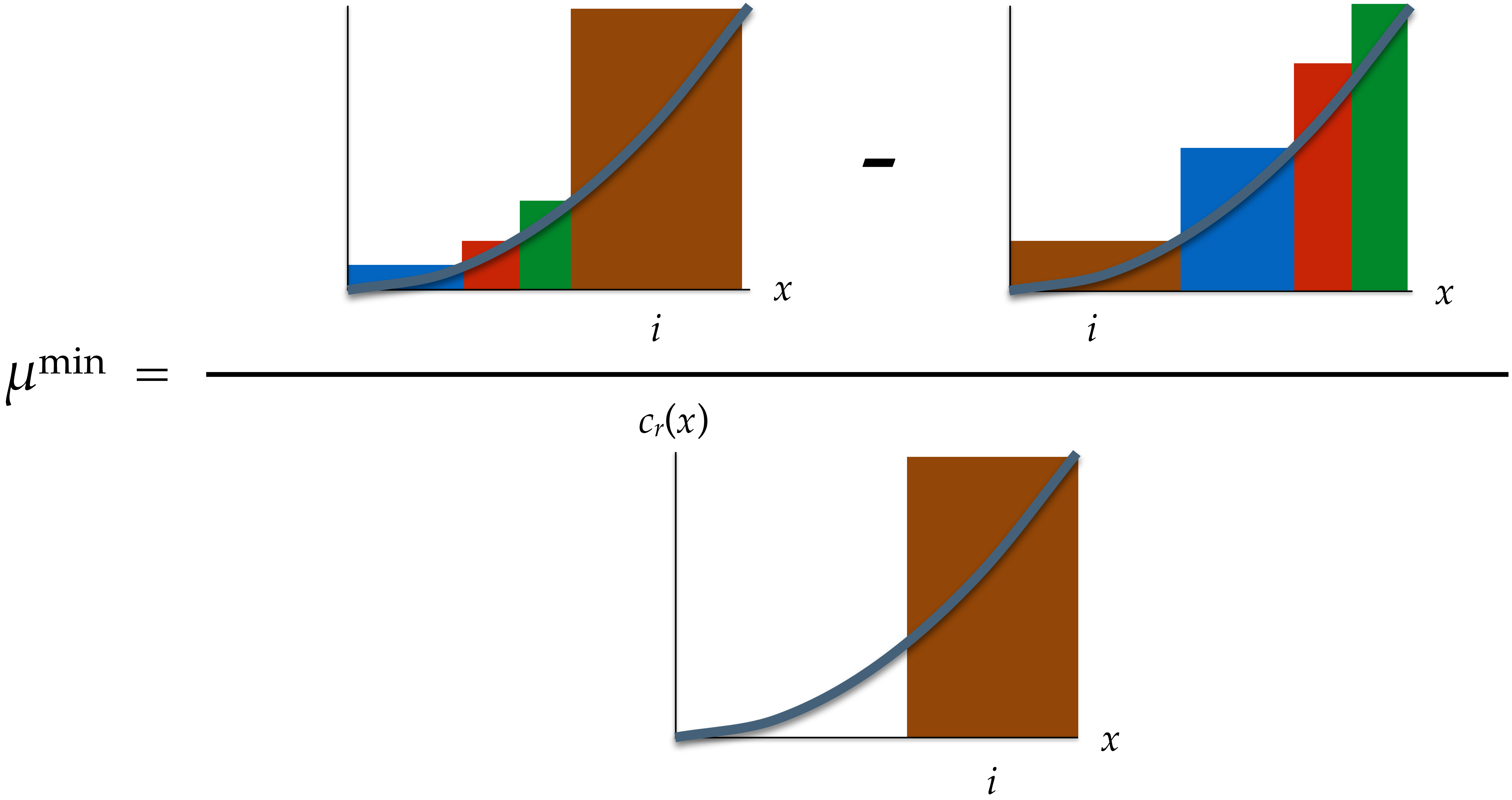
[Hansknecht K. Skopalik, `14]

Theorem Weighted congestion game have an α-approximate pure Nash equilibrium, where $\alpha \leq \min \{1+\mu^{\max}, 1/(1-\mu^{\min})\}$.

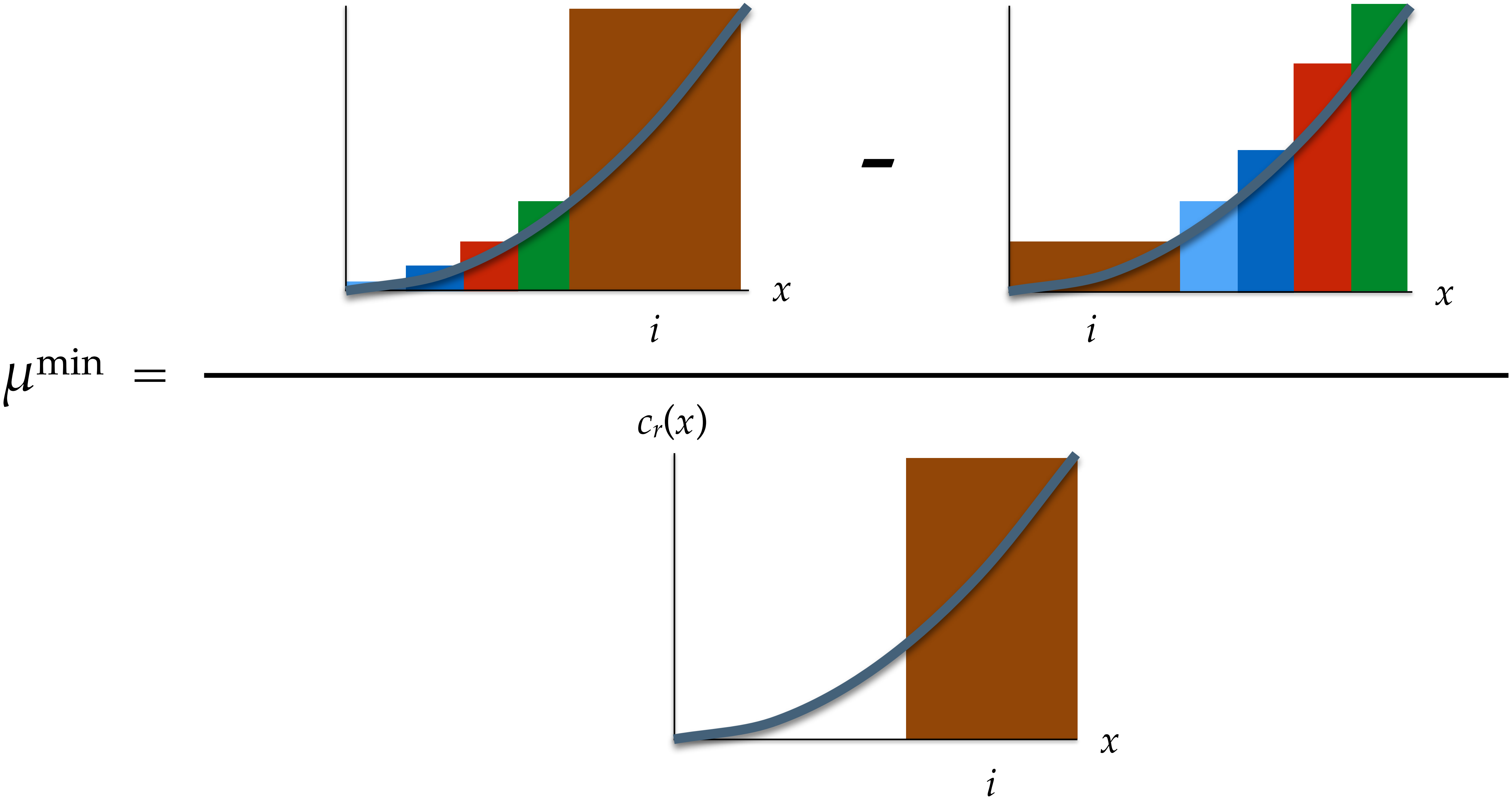
Proof:

- Show that either the maximizing order is an α-approximate potential.

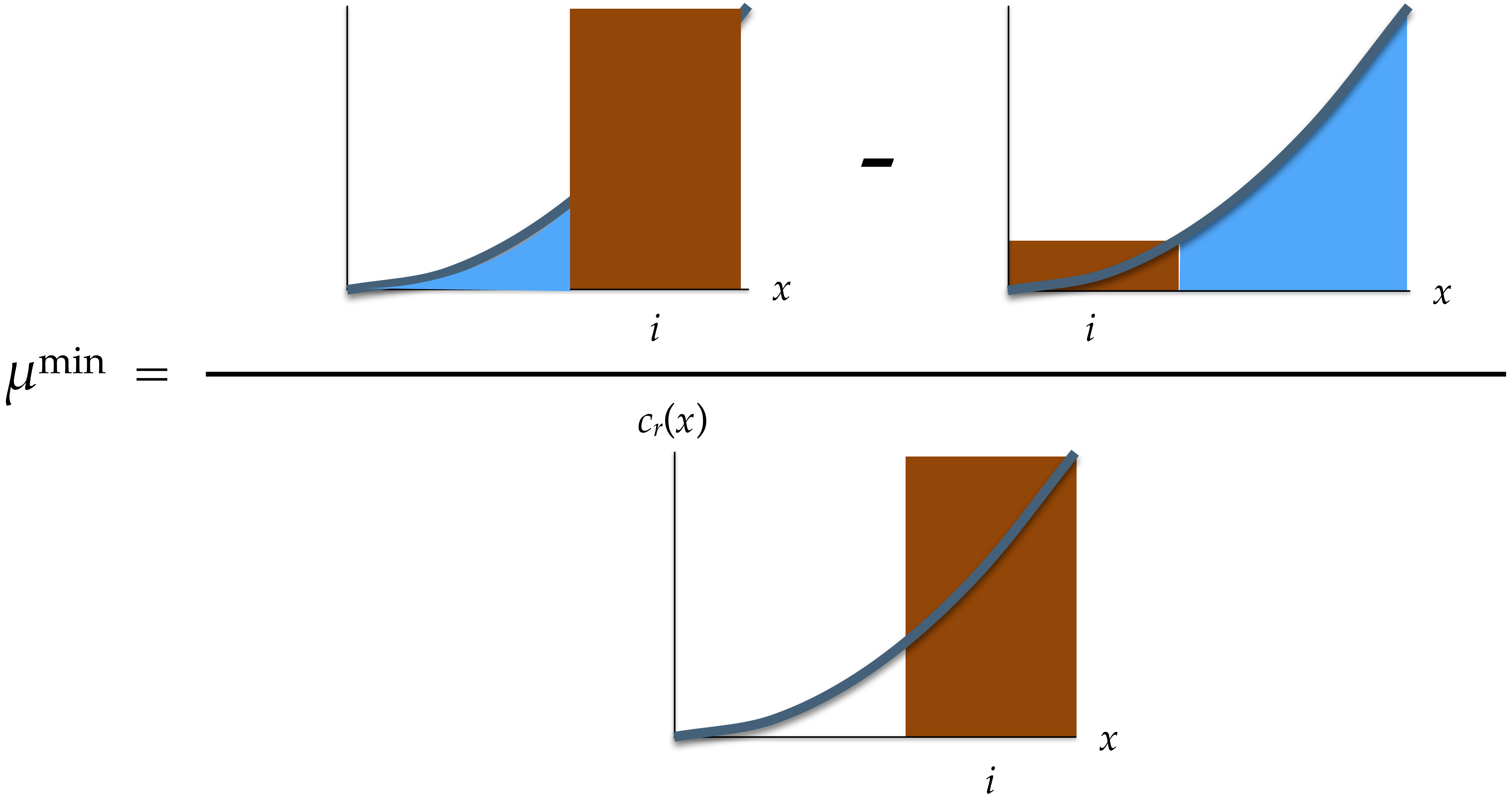
Bounding $\mu^{\min}$



Bounding $\mu^{\min}$



Bounding $\mu^{\min}$



Our results

Functions	Approximation factors	
	2 players	all games
quadratic	$1.054\ldots$	$4/3$
cubic	$1.074\ldots$	$1.785\ldots$
polynomials of max. degree Δ		$\Delta{+}1$
concave		$3/2$

Our results

Functions	Approximation factors	
	2 players	all games
quadratic	$1.054\ldots$	$4/3$
cubic	$1.074\ldots$	$1.785\ldots$
polynomials of max. degree Δ		$\Delta{+}1$
concave		$3/2$

Thank you.